

# Chromatic polynomials and bialgebras of graphs

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## Abstract

The chromatic polynomial is characterized as the unique polynomial invariant of graphs, compatible with two interacting bialgebras structures: the first coproduct is given by partitions of vertices into two parts, the second one by a contraction-extraction process. This gives Hopf-algebraic proofs of Rota's result on the signs of coefficients of chromatic polynomials and of Stanley's interpretation of the values at negative integers of chromatic polynomials. We also give non-commutative version of this construction, replacing graphs by indexed graphs and  $\mathbb{Q}[X]$  by the Hopf algebra **WSym** of set partitions.

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## Introduction

In graph theory, the chromatic polynomial, introduced by Birkhoff and Lewis [3] in order to treat the four color theorem, is a polynomial invariant attached to a graph; its values at  $X = k$  gives the number of valid colorings of the graph with  $k$  colors, for all integer  $k \geq 1$ . Numerous results are known on this object, as for example the alternation signs of the coefficients, a result due to Rota [16], proved with the help of Möbius inversion in certain lattices.

Our aim here is to insert chromatic polynomial into the theory of combinatorial Hopf algebras, and to recover new proofs of these classical results. Our main tools, presented in the first section, will be a Hopf algebra  $(\mathcal{H}_G, m, \Delta)$  and a bialgebra  $(\mathcal{H}_G, m, \delta)$ , both based on graphs. They share the same product, given by disjoint union; the first (cocommutative) coproduct, denoted by  $\Delta$ , is given by partitions of vertices into two parts; the second (not cocommutative) one, denoted by  $\delta$ , is given by a contraction-extraction process. For example:

$$\begin{aligned}\Delta(\nabla) &= \nabla \otimes 1 + 1 \otimes \nabla + 3\mathbf{1} \otimes \bullet + 3\bullet \otimes \mathbf{1}, \\ \delta(\nabla) &= \bullet \otimes \nabla + 3\mathbf{1} \otimes \bullet + \nabla \otimes \dots\end{aligned}$$

$(\mathcal{H}_G, m, \Delta)$  is a Hopf algebra, graded by the cardinality of graphs, and connected, that is to say its connected component of degree 0 is reduced to the base field  $\mathbb{Q}$ : this is what is usually called a *combinatorial Hopf algebra*. On the other side,  $(\mathcal{H}_G, m, \delta)$  is a bialgebra, graded by the degree defined by:

$$\deg(G) = \#\{\text{vertices of } G\} - \#\{\text{connected components of } G\}.$$

These two bialgebras are in cointeraction, a notion described in [5, 8, 13]:  $(\mathcal{H}_G, m, \Delta)$  is a bialgebra-comodule over  $(\mathcal{H}_G, m, \delta)$ , see theorem 7. Another example of interacting bialgebras is the pair  $(\mathbb{Q}[X], m, \Delta)$  and  $(\mathbb{Q}[X], m, \delta)$ , where  $m$  is the usual product of  $\mathbb{Q}[X]$  and the two coproducts  $\Delta$  and  $\delta$  are defined by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \delta(X) = X \otimes X.$$

This has interesting consequences, proved and used on quasi-posets in [8], listed here in theorem 8. In particular:

1. We denote by  $M_G$  the monoid of characters of  $(\mathcal{H}_G, m, \delta)$ . This monoid acts on the set  $E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$  of Hopf algebra morphisms from  $(\mathcal{H}_G, m, \Delta)$  to  $(\mathbb{Q}[X], m, \Delta)$ , via the map:

$$\leftarrow : \begin{cases} E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]} \times M_G & \longrightarrow E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]} \\ (\phi, \lambda) & \longrightarrow \phi \leftarrow \lambda = (\phi \otimes \lambda) \circ \delta. \end{cases}$$

2. There exists a unique  $\phi_0 \in E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$ , homogeneous, such that  $\phi_0(X) = X$ . This morphism is attached to a character  $\lambda_0 \in M_G$ : for any graph  $G$  with  $n$  vertices:

$$\phi_0(G) = \lambda_0(G)X^n.$$

3. There exists a unique  $\phi_1 \in E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$ , compatible with  $m$ ,  $\Delta$  and  $\delta$ . It is given by  $\phi_1 = \phi_0 \leftarrow \lambda_0^{*-1}$ , where  $\lambda_0^{*-1}$  is the inverse of  $\lambda_0$  for the convolution product of  $M_G$ .

The morphisms  $\phi_0$  and  $\phi_1$  and the attached characters are described in the second section. We first prove that, for any graph  $G$ ,  $\lambda_0(G) = 1$ , and  $\phi_1(G)$  is the chromatic polynomial  $P_{chr}(G)$  (proposition 9 and theorem 11). This characterizes the chromatic polynomial as the unique polynomial invariant on graphs compatible with the product  $m$  and both coproducts  $\Delta$  and  $\delta$ . The character attached to the chromatic polynomial is consequently now called the chromatic character and denoted by  $\lambda_{chr}$ . The action of  $M_G$  is used to prove that for any graph  $G$ :

$$P_{chr}(G) = \sum_{\sim} \lambda_{chr}(G| \sim) X^{cl(\sim)},$$

where the sum is over a family of equivalences  $\sim$  on the set of vertices of  $G$ ,  $cl(\sim)$  is the number of equivalence classes of  $\sim$ , and  $G|\sim$  is a graph obtained by restricting  $G$  to the classes of  $\sim$  (corollary 12). Therefore, the knowledge of the chromatic character implies the knowledge of the chromatic polynomial; we give a formula for computing this chromatic character on any graph with the notion (used in Quantum Field Theory) of forests, through the antipode of a quotient of  $(\mathcal{H}_G, m, \delta)$ , see proposition 13. We show how to compute the chromatic polynomial and the chromatic character of a graph by induction on the number of edges by an extraction-contraction of an edge in proposition 15: we obtain an algebraic proof of this classical result, which is classically obtained by a combinatorial study of colorings of  $G$ . As consequences, we obtain proofs of Rota's result on the sign of the coefficients of a chromatic polynomial (corollary 19) and of Stanley's interpretation of values at negative integers of a chromatic polynomial in corollary 24. The link with Rota's proof is made via the lattice attached to a graph, defined in proposition 16.

The last section deals with a non-commutative version of the chromatic polynomial: the Hopf algebra of graphs is replaced by a non-commutative Hopf algebra of indexed graphs, and  $\mathbb{Q}[X]$  is replaced by **WSym**, a Hopf algebra of the Hopf algebra of packed words **WQSym** based on set partitions. For any indexed graph  $G$ , its non-commutative chromatic polynomial  $\mathbf{P}_{chr}(G)$  can also be seen as a symmetric formal series in non-commutative indeterminates: we recover in this way the chromatic symmetric function in noncommuting variables of  $G$ , introduced in [9] and related in [15] to MacMahon symmetric functions. We obtain in this way a non-commutative version of both morphisms  $\phi_0$  (proposition 30) and  $P_{chr}$  (theorem 28), also related by the action of the character  $\lambda_{chr}$ .

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### Notations.

1. All the vector spaces in the text are taken over  $\mathbb{Q}$ .
2. For any integer  $n \geq 0$ , we denote by  $[n]$  the set  $\{1, \dots, n\}$ . In particular,  $[0] = \emptyset$ .
3. The usual product of  $\mathbb{Q}[X]$  is denoted by  $m$ . This algebra is given two bialgebra structures, defined by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \delta(X) = X \otimes X.$$

Identifying  $\mathbb{Q}[X, Y]$  and  $\mathbb{Q}[X] \otimes \mathbb{Q}[Y]$ , for any  $P \in \mathbb{Q}[X]$ :

$$\Delta(P)(X, Y) = P(X + Y), \quad \delta(P)(X, Y) = P(XY).$$

The counit of  $\Delta$  is given by:

$$\forall P \in \mathbb{Q}[X], \varepsilon(P) = P(0).$$

The counit of  $\delta$  is given by:

$$\forall P \in \mathbb{Q}[X], \varepsilon'(P) = P(1).$$

# 1 Hopf algebraic structures on graphs

We refer to [10] for classical results and vocabulary on graphs. Recall that a graph is a pair  $G = (V(G), E(G))$ , where  $V(G)$  is a finite set, and  $E(G)$  is a subset of the set of parts of  $V(G)$  of cardinality 2. In sections 1 and 2, we shall work with isoclasses of graphs, which we will simply call graphs. The set of graphs is denoted by  $\mathcal{G}$ . For example, here are graphs of cardinality  $\leq 4$ :

$$1; \quad \cdot; \quad \cdot, \dots; \quad \nabla, \vee, \cdot, \dots; \quad \boxtimes, \boxdot, \boxminus, \boxplus, \ltimes, \rtimes, \nabla \cdot, \vee \cdot, \cdot \cdot, \cdot \cdot, \dots$$

For any graph  $G$ , we denote by  $|G|$  the cardinality of  $G$  and by  $cc(G)$  the number of its connected components.

A graph is *totally disconnected* if it has no edge.

We denote by  $\mathcal{H}_{\mathcal{G}}$  the vector space generated by the set of graphs. The disjoint union of graphs gives it a commutative, associative product  $m$ . As an algebra,  $\mathcal{H}_{\mathcal{G}}$  is (isomorphic to) the free commutative algebra generated by connected graphs.

## 1.1 The first coproduct

**Definition 1** Let  $G$  be a graph and  $I \subseteq V(G)$ . The graph  $G|_I$  is defined by:

- $V(G|_I) = I$ .
- $E(G|_I) = \{\{x, y\} \in E(G) \mid x, y \in I\}$ .

We refer to [1, 12, 19] for classical results and notations on bialgebras and Hopf algebras. The following Hopf algebra is introduced in [17]:

**Proposition 2** We define a coproduct  $\Delta$  on  $\mathcal{H}_{\mathcal{G}}$  by:

$$\forall G \in \mathcal{G}, \Delta(G) = \sum_{V(G)=I \sqcup J} G|_I \otimes G|_J.$$

Then  $(\mathcal{H}_{\mathcal{G}}, m, \Delta)$  is a graded, connected, cocommutative Hopf algebra. Its counit is given by:

$$\forall G \in \mathcal{G}, \varepsilon(G) = \delta_{G,1}.$$

**Proof.** If  $G, H$  are two graphs, then  $V(GH) = V(G) \sqcup V(H)$ , so:

$$\begin{aligned} \Delta(GH) &= \sum_{\substack{V(G)=I \sqcup J, \\ V(H)=K \sqcup L}} GH|_{I \sqcup K} \otimes GH|_{J \sqcup L} \\ &= \sum_{\substack{V(G)=I \sqcup J, \\ V(H)=K \sqcup L}} G|_I H|_K \otimes G|_J H|_L \\ &= \Delta(G) \Delta(H). \end{aligned}$$

If  $G$  is a graph, and  $I \subseteq J \subseteq V(G)$ , then  $(G|_I)|_J = G|_J$ . Hence:

$$\begin{aligned} (\Delta \otimes Id) \circ \Delta(G) &= \sum_{\substack{V(G)=I \sqcup L, \\ I=J \sqcup K}} (G|_I)|_J \otimes (G|_I)|_K \otimes G|_L \\ &= \sum_{V(G)=J \sqcup K \sqcup L} G|_J \otimes G|_K \otimes G|_L \\ &= \sum_{\substack{V(G)=J \sqcup I, \\ I=K \sqcup L}} G|_J \otimes (G|_I)|_K \otimes (G|_I)|_L \\ &= (Id \otimes \Delta) \circ \Delta(G). \end{aligned}$$

So  $\Delta$  is coassociative. It is obviously cocommutative. □

### Examples.

$$\begin{aligned}\Delta(\cdot) &= \cdot \otimes 1 + 1 \otimes \cdot, \\ \Delta(\mathbf{1}) &= \mathbf{1} \otimes 1 + 1 \otimes \mathbf{1} + 2 \cdot \otimes \cdot, \\ \Delta(\nabla) &= \nabla \otimes 1 + 1 \otimes \nabla + 3\mathbf{1} \otimes \cdot + 3 \cdot \otimes \mathbf{1}, \\ \Delta(\mathbf{\nabla}) &= \mathbf{\nabla} \otimes 1 + 1 \otimes \mathbf{\nabla} + 2\mathbf{1} \otimes \cdot + \cdot \otimes \mathbf{1} + 2 \cdot \otimes \mathbf{1} + \cdot \otimes \dots\end{aligned}$$

## 1.2 The second coproduct

**Notations.** Let  $V$  be a finite set  $\sim$  be an equivalence on  $V$ .

- We denote by  $\pi_\sim : V \longrightarrow V/\sim$  the canonical surjection.
- We denote by  $cl(\sim)$  the cardinality of  $V/\sim$ .

**Definition 3** Let  $G$  a graph, and  $\sim$  be an equivalence relation on  $V(G)$ .

1. (Contraction). The graph  $V(G)/\sim$  is defined by:

$$\begin{aligned}V(G/\sim) &= V(G)/\sim, \\ E(G/\sim) &= \{\{\pi_\sim(x), \pi_\sim(y)\} \mid \{x, y\} \in E(G), \pi_\sim(x) \neq \pi_\sim(y)\}.\end{aligned}$$

2. (Extraction). The graph  $V(G)|\sim$  is defined by:

$$\begin{aligned}V(G|\sim) &= V(G), \\ E(G|\sim) &= \{\{x, y\} \in E(G) \mid x \sim y\}.\end{aligned}$$

3. We shall write  $\sim \triangleleft G$  if, for any  $x \in V(G)$ ,  $G|_{\pi_\sim(x)}$  is connected.

Roughly speaking,  $G/\sim$  is obtained by contracting each equivalence class of  $\sim$  to a single vertex, and by deleting the loops and multiple edges created in the process;  $G|\sim$  is obtained by deleting the edges which extremities are not equivalent, so is the product of the restrictions of  $G$  to the equivalence classes of  $\sim$ .

We now define a coproduct on  $\mathcal{H}_G$ . A similar construction is defined on various families of oriented graphs in [13].

**Proposition 4** We define a coproduct  $\delta$  on  $\mathcal{H}_G$  by:

$$\forall G \in \mathcal{G}, \delta(G) = \sum_{\sim \triangleleft G} (G/\sim) \otimes (G|\sim).$$

Then  $(\mathcal{H}_G, m, \delta)$  is a bialgebra. Its counit is given by:

$$\forall G \in \mathcal{G}, \varepsilon'(G) = \begin{cases} 1 & \text{if } G \text{ is totally disconnected,} \\ 0 & \text{otherwise.} \end{cases}$$

It is graded, putting:

$$\forall G \in \mathcal{G}, \deg(G) = |G| - cc(G).$$

**Proof.** Let  $G, H$  be graphs and  $\sim$  be an equivalence on  $V(GH) = V(G) \sqcup V(H)$ . We put  $\sim' = \sim|_{V(G)}$  and  $\sim'' = \sim|_{V(H)}$ . The connected components of  $GH$  are the ones of  $G$  and  $H$ , so  $\sim \triangleleft GH$  if, and only if, the two following conditions are satisfied:

- $\sim' \triangleleft G$  and  $\sim'' \triangleleft H$ .
- If  $x \sim y$ , then  $(x, y) \in V(G)^2 \sqcup V(H)^2$ .

Note that the second point implies that  $\sim$  is entirely determined by  $\sim'$  and  $\sim''$ . Moreover, if this holds,  $(GH)/\sim = (G/\sim')(H/\sim'')$  and  $(GH)|\sim = (G|\sim')(H|\sim'')$ , so:

$$\delta(GH) = \sum_{\sim' \triangleleft G, \sim'' \triangleleft H} (G/\sim')(H/\sim'') \otimes (G|\sim')(H|\sim'') = \delta(G)\delta(H).$$

Let  $G$  be a graph. If  $\sim \triangleleft G$ , the connected components of  $G/\sim$  are the image by the canonical surjection of the connected components of  $G$ ; the connected components of  $G|\sim$  are the equivalence classes of  $\sim$ . If  $\sim$  and  $\sim'$  are two equivalences on  $G$ , we shall denote  $\sim' \leq \sim$  if for all  $x, y \in V(G)$ ,  $x \sim' y$  implies  $x \sim y$ . Then:

$$\begin{aligned} (\delta \otimes Id) \circ \delta(G) &= \sum_{\sim \triangleleft G, \sim' \triangleleft G/\sim} (G/\sim)/\sim' \otimes (G/\sim)|\sim' \otimes G|\sim \\ &= \sum_{\substack{\sim, \sim' \triangleleft G, \\ \sim' \leq \sim}} (G/\sim)/\sim' \otimes (G/\sim)|\sim' \otimes G|\sim \\ &= \sum_{\substack{\sim, \sim' \triangleleft G, \\ \sim' \leq \sim}} (G/\sim') \otimes (G|\sim')/\sim \otimes (G|\sim')|\sim \\ &= \sum_{\sim \triangleleft G, \sim' \triangleleft G|\sim} (G/\sim') \otimes (G|\sim')/\sim \otimes (G|\sim')|\sim \\ &= (Id \otimes \delta) \circ \delta(G). \end{aligned}$$

So  $\delta$  is coassociative.

We define two special equivalence relations  $\sim_0$  and  $\sim_1$  on  $G$ : for all  $x, y \in V(G)$ ,

- $x \sim_0 y$  if, and only if,  $x = y$ .
- $x \sim_1 y$  if, and only if,  $x$  and  $y$  are in the same connected component of  $G$ .

Note that  $\sim_0, \sim_1 \triangleleft G$ . Moreover, if  $\sim \triangleleft G$ ,  $G/\sim$  is not totally disconnected, except if  $\sim = \sim_1$ ;  $G|\sim$  is not totally disconnected, except if  $\sim = \sim_0$ . Hence:

- If  $G$  is totally disconnected, then  $\delta(G) = G \otimes G$ .
- Otherwise, denoting by  $n$  the degree of  $G$  and by  $k$  its number of connected components:

$$\delta(G) = \cdot^k \otimes G + G \otimes \cdot^n + Ker(\varepsilon') \otimes Ker(\varepsilon').$$

So  $\varepsilon'$  is indeed the counit of  $\delta$ .

Let  $G$  be a graph, with  $n$  vertices and  $k$  connected components (so of degree  $n - k$ ). Let  $\sim \triangleleft G$ . Then:

1.  $G/\sim$  has cardinality  $cl(\sim)$  and  $k$  connected components, so is of degree  $cl(\sim) - k$ .
2.  $G|\sim$  has cardinality  $n$  and  $cl(\sim)$  connected components, so is of degree  $n - cl(\sim)$ .

Hence,  $\deg(G/\sim) + \deg(G|\sim) = cl(\sim) - k + n - cl(\sim) = n - k = \deg(G)$ :  $\delta$  is homogeneous.  $\square$

**Examples.**

$$\begin{aligned}\delta(\bullet) &= \bullet \otimes \bullet, & \delta(\mathbf{1}) &= \bullet \otimes \mathbf{1} + \mathbf{1} \otimes \bullet, \\ \delta(\nabla) &= \bullet \otimes \nabla + 3\mathbf{1} \otimes \bullet + \nabla \otimes \bullet, & \delta(\mathbf{V}) &= \bullet \otimes \mathbf{V} + 2\mathbf{1} \otimes \bullet + \mathbf{V} \otimes \bullet.\end{aligned}$$

**Remark.** Let  $G \in \mathcal{G}$ . The following conditions are equivalent:

- $\varepsilon'(G) = 1$ .
- $\varepsilon'(G) \neq 0$ .
- $\deg(G) = 0$ .
- $G$  is totally disconnected.

### 1.3 Antipode

$(\mathcal{H}_G, m, \delta)$  is not a Hopf algebra: the group-like element  $\bullet$  has no inverse. However, the graduation of  $(\mathcal{H}_G, m, \delta)$  induced a graduation of  $\mathcal{H}'_G = (\mathcal{H}_G, m, \delta)/\langle \bullet - 1 \rangle$ , which becomes a graded, connected bialgebra, hence a Hopf algebra; we denote its antipode by  $S'$ . Note that, as a commutative algebra,  $\mathcal{H}'_G$  is freely generated by connected graphs different from  $\bullet$ .

The notations and ideas of the following definition and theorem come from Quantum Field Theory, where they are applied to Renormalization with the help of Hopf algebras of Feynman graphs; see for example [6, 7] for an introduction.

**Definition 5** Let  $G$  be a connected graph,  $G \neq \bullet$ .

1. A forest of  $G$  is a set  $\mathcal{F}$  of subsets of  $V(G)$ , such that:

- (a)  $V(G) \in \mathcal{F}$ .
- (b) If  $I, J \in \mathcal{F}$ , then  $I \subseteq J$ , or  $J \subseteq I$ , or  $I \cap J = \emptyset$ .
- (c) For all  $I \in \mathcal{F}$ ,  $G|_I$  is connected.

The set of forests of  $G$  is denoted by  $\mathbb{F}(G)$ .

2. Let  $\mathcal{F} \in \mathbb{F}(G)$ ; it is partially ordered by the inclusion. For any  $I \in \mathbb{F}(G)$ , the relation  $\sim_I$  is the equivalence on  $I$  which classes are the maximal elements (for the inclusion) of  $\{J \in \mathcal{F} \mid J \subsetneq I\}$  (if this is non-empty), and singletons. We put:

$$G_{\mathcal{F}} = \prod_{I \in \mathcal{F}} (G|_I) / \sim_I.$$

**Examples.** The graph  $\mathbf{1}$  has only one forest,  $\mathcal{F} = \{\mathbf{1}\}$ ;  $\mathbf{1}_{\mathcal{F}} = \mathbf{1}$ . The graph  $\nabla$  has four forests:

- $\mathcal{F} = \{\nabla\}$ ; in this case,  $\nabla_{\mathcal{F}} = \nabla$ .
- Three forests  $\mathcal{F} = \{\nabla, \mathbf{1}\}$ ; for each of them,  $\nabla_{\mathcal{F}} = \mathbf{1}\mathbf{1}$ .

**Theorem 6** For any connected graph  $G$ ,  $G \neq \bullet$ , in  $\mathcal{H}'_G$ :

$$S(G) = \sum_{\mathcal{F} \in \mathbb{F}(G)} (-1)^{\#\mathcal{F}} G_{\mathcal{F}}.$$

**Proof.** By induction on the number  $n$  of vertices of  $G$ . If  $n = 2$ , then  $G = \mathbf{1}$ . As  $\delta'(\mathbf{1}) = \mathbf{1} \otimes 1 + 1 \otimes \mathbf{1}$ ,  $S'(\mathbf{1}) = -\mathbf{1} = -\mathbf{1}_{\mathcal{F}}$ , where  $\mathcal{F} = \{\mathbf{1}\}$  is the unique forest of  $\mathbf{1}$ . Let us assume the result at all ranks  $< n$ . Then:

$$\begin{aligned}
S'(G) &= -G - \sum_{\sim \triangleleft G} (G/\sim) S'(G|\sim) \\
&= -G - \sum_{\substack{\sim \triangleleft G, \\ G/\sim = \{I_1, \dots, I_k\}}} \sum_{\mathcal{F}_i \in \mathbb{F}(G|_{I_i})} (-1)^{\sharp \mathcal{F}_1 + \dots + \sharp \mathcal{F}_k} (G/\sim)_{\mathcal{F}_1} \dots (G|_{I_1})_{\mathcal{F}_1} \\
&= -G - \sum_{\mathcal{F} \in \mathbb{F}(G), \mathcal{F} \neq \{G\}} (-1)^{\sharp \mathcal{F} - 1} G_{\mathcal{F}} \\
&= \sum_{\mathcal{F} \in \mathbb{F}(G)} (-1)^{\sharp \mathcal{F}} G_{\mathcal{F}}.
\end{aligned}$$

For the third equality,  $\mathcal{F} = \{G\} \sqcup \mathcal{F}_1 \sqcup \dots \sqcup \mathcal{F}_k$ . □

## 1.4 Cointeraction

**Theorem 7** *With the coaction  $\delta$ ,  $(\mathcal{H}_{\mathcal{G}}, m, \Delta)$  and  $(\mathcal{H}_{\mathcal{G}}, m, \delta)$  are in cointeraction, that is to say that  $(\mathcal{H}_{\mathcal{G}}, m, \Delta)$  is a  $(\mathcal{H}_{\mathcal{G}}, m, \delta)$ -comodule bialgebra, or a Hopf algebra in the category of  $(\mathcal{H}_{\mathcal{G}}, m, \delta)$ -comodules. In other words:*

- $\delta(1) = 1 \otimes 1$ .
- $m_{2,4}^3 \circ (\delta \otimes \delta) \circ \Delta = (\Delta \otimes Id) \circ \delta$ , with:

$$m_{2,4}^3 : \begin{cases} \mathcal{H}_{\mathcal{G}} \otimes \mathcal{H}_{\mathcal{G}} \otimes \mathcal{H}_{\mathcal{G}} \otimes \mathcal{H}_{\mathcal{G}} & \longrightarrow & \mathcal{H}_{\mathcal{G}} \otimes \mathcal{H}_{\mathcal{G}} \otimes \mathcal{H}_{\mathcal{G}} \\ a_1 \otimes b_1 \otimes a_2 \otimes b_2 & \longrightarrow & a_1 \otimes a_2 \otimes b_1 b_2. \end{cases}$$

- For all  $a, b \in \mathcal{H}_{\mathcal{G}}$ ,  $\delta(ab) = \delta(a)\delta(b)$ .
- For all  $a \in \mathcal{H}_{\mathcal{G}}$ ,  $(\varepsilon \otimes Id) \circ \delta(a) = \varepsilon(a)1$ .

**Proof.** The first and third points are already proved, and the fourth one is immediate when  $a \in \mathcal{G}$ . Let us prove the second point. For any graphs  $G, H$ :

$$\begin{aligned}
(\Delta \otimes Id) \circ \delta(GH) &= \sum_{\sim \triangleleft G, V(G)/\sim = I \sqcup J} (G/\sim)_{|I} \otimes (G/\sim)_{|J} \otimes G|\sim \\
&= \sum_{\substack{V(G) = I' \sqcup J', \\ \sim' \triangleleft G|_{I'}, \sim'' \triangleleft G|_J}} (G|_{I'})/\sim' \otimes (G|_{J'})/\sim'' \otimes (G|_{I'})|\sim' (G|_{J'})|\sim'' \\
&= m_{2,4}^3 \circ (\delta \otimes \delta) \circ \Delta(G).
\end{aligned}$$

For the second equality,  $I' = \pi_{\sim}^{-1}(I)$ ,  $I'' = \pi_{\sim}^{-1}(J)$ ,  $\sim' = \sim|_{I'}$  and  $\sim'' = \sim|_{J'}$ . □

We can apply the results of [8]:

**Theorem 8** *We denote by  $M_{\mathcal{G}}$  the monoid of characters of  $\mathcal{H}_{\mathcal{G}}$ .*

1. Let  $\lambda \in M_{\mathcal{G}}$ . It is an invertible element if, and only if,  $\lambda(\cdot) \neq 0$ .
2. Let  $B$  be a Hopf algebra and  $E_{\mathcal{H}_{\mathcal{G}} \rightarrow B}$  be the set of Hopf algebra morphisms from  $\mathcal{H}_{\mathcal{G}}$  to  $B$ . Then  $M_{\mathcal{G}}$  acts on  $E_{\mathcal{H}_{\mathcal{G}} \rightarrow B}$  by:

$$\leftarrow : \begin{cases} E_{\mathcal{H}_{\mathcal{G}} \rightarrow B} \times M_{\mathcal{G}} & \longrightarrow & E_{\mathcal{H}_{\mathcal{G}} \rightarrow B} \\ (\phi, \lambda) & \longrightarrow & \phi \leftarrow \lambda = (\phi \otimes \lambda) \circ \delta. \end{cases}$$



3. There exists a unique  $\phi_0 \in E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$ , homogeneous, such that  $\phi_0(\cdot) = X$ ; there exists a unique  $\lambda_0 \in M_G$  such that:

$$\forall G \in \mathcal{G}, \phi_0(G) = \lambda_0(G)X^{|G|}.$$

Moreover, the following map is a bijection:

$$\begin{cases} M_G & \longrightarrow E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]} \\ \lambda & \longrightarrow \phi_0 \leftarrow \lambda. \end{cases}$$

4. Let  $\lambda \in M_G$ . There exists a unique element  $\phi \in E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$  such that:

$$\forall x \in \mathcal{H}_G, \phi(x)(1) = \lambda(x).$$

This morphism is  $\phi_0 \leftarrow (\lambda_0^{*-1} * \lambda)$ .

5. There exists a unique morphism  $\phi_1 : \mathcal{H}_G \longrightarrow \mathbb{Q}[X]$ , such that:

- $\phi_1$  is a Hopf algebra morphism from  $(\mathcal{H}_G, m, \Delta)$  to  $(\mathbb{Q}[X], m, \Delta)$ .
- $\phi_1$  is a bialgebra morphism from  $(\mathcal{H}_G, m, \delta)$  to  $(\mathbb{Q}[X], m, \delta)$ .

This morphism is the unique element of  $E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$  such that:

$$\forall x \in \mathcal{H}_G, \phi_1(x)(1) = \varepsilon'(x).$$

Moreover,  $\phi_1 = \phi_0 \leftarrow \lambda_0^{*-1}$ .

We shall determine  $\phi_0$  and  $\phi_1$  in the next section.

## 2 The chromatic polynomial as a Hopf algebra morphism

### 2.1 Determination of $\phi_0$

**Proposition 9** For any graph  $G$ :

$$\phi_0(G) = X^{|G|}, \quad \lambda_0(G) = 1.$$

**Proof.** Let  $\psi : \mathcal{H}_G \longrightarrow \mathbb{Q}[X]$ , sending any graph  $G$  to  $X^{|G|}$ . It is a homogeneous algebra morphism. For any graph  $G$ , of degree  $n$ :

$$(\psi \otimes \psi) \circ \Delta(G) = \sum_{V(G)=I \sqcup J} X^{|I|} \otimes X^{|J|} = \sum_{i=0}^n \binom{n}{i} X^i \otimes X^{n-i} = \Delta(X^n) = \Delta \circ \psi(G).$$

So  $\psi$  is a Hopf algebra morphism. As  $\psi(\cdot) = X$ ,  $\psi = \phi_0$ . □

### 2.2 Determination of $\phi_1$

Let us recall the definition of the chromatic polynomial, due to Birkhoff and Lewis [3]:

**Definition 10** Let  $G$  be a graph and  $X$  a set.

1. A  $X$ -coloring of  $G$  is a map  $f : V(G) \longrightarrow X$ .
2. A  $\mathbb{N}$ -coloring of  $G$  is packed if  $f(V(G)) = [k]$ , with  $k \geq 0$ . The set of packed coloring of  $G$  is denoted by  $\mathbb{PC}(G)$ .

3. A valid  $X$ -coloring of  $G$  by  $X$  is a  $X$ -coloring  $f$  such that if  $\{i, j\} \in E(G)$ , then  $f(i) \neq f(j)$ . The set of valid  $X$ -colorings of  $G$  is denoted by  $\mathbb{VC}(G, X)$ ; the set of packed valid colorings of  $G$  is denoted by  $\mathbb{PVC}(G)$ .
4. An independent subset of  $G$  is a subset  $I$  of  $V(G)$  such that  $G|_I$  is totally disconnected. We denote by  $\mathbb{IP}(G)$  the set of partitions  $\{I_1, \dots, I_k\}$  of  $V(G)$  such that for all  $p \in [k]$ ,  $I_p$  is an independent subset of  $G$ .
5. For any  $k \geq 1$ , the number of valid  $[k]$ -colorings of  $G$  is denoted by  $P_{chr}(G)(k)$ . This defines a unique polynomial  $P_{chr}(G) \in \mathbb{Q}[X]$ , called the chromatic polynomial of  $G$ .

Note that if  $f$  is a  $X$ -coloring of a graph  $G$ , it is valid if, and only if, the partition of  $V(G)$   $\{f^{-1}(x) \mid x \in f(V(G))\}$  belongs to  $\mathbb{IP}(G)$ .

**Theorem 11** *The morphism  $P_{chr} : \mathcal{H}_G \longrightarrow \mathbb{Q}[X]$  is the morphism  $\phi_1$  of theorem 8.*

**Proof.** It is immediate that, for any graphs  $G$  and  $H$ ,  $P_{chr}(GH)(k) = P_{chr}(G)(k)P_{chr}(H)(k)$  for any  $k$ , so  $P_{chr}(GH) = P_{chr}(G)P_{chr}(H)$ :  $P_{chr}$  is an algebra morphism. Let  $G$  be a graph, and  $k, l \geq 1$ . We consider the two sets:

$$\begin{aligned} C &= \mathbb{VC}(G, [k+l]), \\ D &= \{(I, c', c'') \mid I \subseteq V(G), c' \in \mathbb{VC}(G|_I, [k]), c'' \in \mathbb{VC}(G|_{V(G) \setminus I}, [l])\}. \end{aligned}$$

We define a map  $\theta : C \longrightarrow D$  by  $\theta(c) = (I, c', c'')$ , with:

- $I = \{x \in V(G) \mid c(x) \in [k]\}$ .
- For all  $x \in I$ ,  $c'(x) = c(x)$ .
- For all  $x \notin I$ ,  $c''(x) = c(x) - k$ .

We define a map  $\theta' : D \longrightarrow C$  by  $\theta'(I, c', c'') = c$ , with:

- For all  $x \in I$ ,  $c(x) = c'(x)$ .
- For all  $x \notin I$ ,  $c(x) = c''(x) + k$ .

Both  $\theta$  and  $\theta'$  are well-defined; moreover,  $\theta \circ \theta' = Id_D$  and  $\theta' \circ \theta = Id_C$ , so  $\theta$  is a bijection. Via the identification of  $\mathbb{Q}[X] \otimes \mathbb{Q}[X]$  and  $\mathbb{Q}[X, Y]$ :

$$\begin{aligned} \Delta \circ P_{chr}(G)(k, l) &= P_{chr}(G)(k+l) \\ &= \sharp C \\ &= \sharp D \\ &= \sum_{I \subseteq V(G)} P_{chr}(G|_I)(k) P_{chr}(G|_{V(G) \setminus I})(l) \\ &= (P_{chr} \otimes P_{chr}) \left( \sum_{V(G) = I \sqcup J} G|_I \otimes G|_J \right) (k, l) \\ &= (P_{chr} \otimes P_{chr}) \circ \Delta(G)(k, l). \end{aligned}$$

As this is true for all  $k, l \geq 1$ ,  $\Delta \circ P_{chr}(G) = (P_{chr} \otimes P_{chr}) \circ \Delta(G)$ . Moreover:

$$\varepsilon(G) = \varepsilon \circ P_{chr}(G) = P_{chr}(G)(0) = \begin{cases} 1 & \text{if } G \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

So  $P_{chr} \in E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$ . For any graph  $G$ :

$$\begin{aligned} P_{chr}(G)(1) &= \begin{cases} 1 & \text{if } G \text{ is totally disconnected,} \\ 0 & \text{otherwise;} \end{cases} \\ &= \varepsilon'(G). \end{aligned}$$

So  $\phi_1 = P_{chr}$ . □

**Corollary 12** *For any connected graph  $G$ , we put:*

$$\lambda_{chr}(G) = \frac{dP_{chr}(G)}{dX}(0).$$

*We extend  $\lambda$  as an element of  $M_G$ : for any graph  $G$ , if  $G_1, \dots, G_k$  are the connected components of  $G$ ,*

$$\lambda_{chr}(G) = \lambda_{chr}(G_1) \dots \lambda_{chr}(G_k).$$

*Then  $\lambda_{chr}$  is an invertible element of  $M_G$ , and  $\lambda_{chr}^{*-1} = \lambda_0$ . Moreover:*

$$\forall G \in \mathcal{G}, P_{chr}(G) = \sum_{\sim \triangleleft G} \lambda_{chr}(G| \sim) X^{cl(\sim)}.$$

**Proof.** By theorem 8, there exists a unique  $\lambda_{chr} \in M_G$ , such that  $P_{chr} = \phi_0 \leftarrow \lambda_{chr}$ . For any graph  $G$ :

$$P_{chr}(G) = (\phi_0 \otimes \lambda_{chr}) \circ \delta(G) = \sum_{\sim \triangleleft G} \phi_0(G/ \sim) \lambda_{chr}(G| \sim) = \sum_{\sim \triangleleft G} X^{cl(\sim)} \lambda_{chr}(G| \sim).$$

If  $G$  is connected, there exists a unique  $\sim \triangleleft G$  such that  $cl(\sim) = 1$ : this is the equivalence relation such that for any  $x, y \in V(G)$ ,  $x \sim y$ . Hence, the coefficient of  $X$  in  $P_{chr}(X)$  is  $\lambda(G| \sim) = \lambda(G)$ , so:

$$\lambda_{chr}(G) = \frac{dP_{chr}(G)}{dX}(0).$$

By theorem 8,  $\lambda_{chr} = \lambda_0^{*-1}$ . □

The character  $\lambda_{chr}$  will be called the *chromatic character*.

**Proposition 13**  $\lambda_{chr}(\bullet) = 1$ ; if  $G$  is a connected graph,  $G \neq \bullet$ , then:

$$\lambda_{chr}(G) = \sum_{\mathcal{F} \in \mathbb{F}(G)} (-1)^{\sharp \mathcal{F}}.$$

**Proof.** We have  $\lambda_{chr}(\bullet) = \lambda_{chr}^{*-1}(\bullet) = 1$ , so both  $\lambda_{chr}$  and  $\lambda_{chr}^{*-1}$  can be seen as characters on  $\mathcal{H}'_G$ . Hence, for any connected graph  $G$ , different from  $\bullet$ :

$$\lambda_{chr}(G) = \lambda_{chr}^{*-1} \circ S'(G) = \sum_{\mathcal{F} \in \mathbb{F}(G)} (-1)^{\sharp \mathcal{F}} \lambda_{chr}^{*-1}(G_{\mathcal{F}}) = \sum_{\mathcal{F} \in \mathbb{F}(G)} (-1)^{\sharp \mathcal{F}},$$

as  $\lambda_{chr}^{*-1}(H) = 1$  for any graph  $H$ . □

**Examples.**

1.

$G$	$\bullet$	$\mathbf{1}$	$\nabla$	$\mathbf{V}$	$\boxtimes$	$\boxminus$	$\sqcup$	$\square$	$\bowtie$	$\sqcup$
$\lambda_{chr}(G)$	1	-1	2	1	-6	-4	-2	-3	-1	-1

2. If  $G$  is a complete graph with  $n$  vertices,  $P_{chr}(G)(X) = X(X-1)\dots(X-n+1)$ , so  $\lambda_{chr}(G) = (-1)^{n-1}(n-1)!$ .

## 2.3 Extraction and contraction of edges

**Definition 14** Let  $G$  be a graph and  $e \in E(G)$ .

1. (Contraction of  $e$ ). The graph  $G/e$  is  $G/\sim_e$ , where  $\sim_e$  is the equivalence which classes are  $e$  and singletons.
2. (Subtraction of  $e$ ). The graph  $G \setminus e$  is the graph  $(V(G), E(G) \setminus \{e\})$ .
3. We shall say that  $e$  is a bridge (or an isthmus) of  $G$  if  $cc(G \setminus e) > cc(G)$ .

We now give an algebraic proof of the following well-known result:

**Proposition 15** For any graph  $G$ , for any edge  $e$  of  $G$ :

$$P_{chr}(G) = P_{chr}(G \setminus e) - P_{chr}(G/e);$$

$$\lambda_{chr}(G) = \begin{cases} -\lambda_{chr}(G/e) & \text{if } e \text{ is a bridge,} \\ \lambda_{chr}(G \setminus e) - \lambda_{chr}(G/e) & \text{otherwise.} \end{cases}$$

**Proof.** Let  $G$  be a graph, and  $e \in E(G)$ . Let us prove that for all  $k \geq 1$ ,  $P_{chr}(G)(k) = P_{chr}(G \setminus e)(k) - P_{chr}(G/e)(k)$ . We proceed by induction on  $k$ . If  $k = 1$ ,  $P_{chr}(G)(1) = \varepsilon'(G) = 0$ . If  $G$  has only one edge, then  $G \setminus e$  and  $G/e$  are totally disconnected, and:

$$P_{chr}(G \setminus e)(1) - P_{chr}(G/e)(1) = 1 - 1 = 0.$$

Otherwise,  $G \setminus e$  and  $G/e$  have edges, and:

$$P_{chr}(G \setminus e)(1) - P_{chr}(G/e)(1) = 0 - 0 = 0.$$

Let us assume the result at rank  $k$ . Putting  $e = \{x, y\}$ :

$$\begin{aligned}
& P_{chr}(G \setminus e)(k+1) - P_{chr}(G/e)(k+1) \\
&= \sum_{V(G)=I \sqcup J} P_{chr}((G \setminus e)|_I)(k) P_{chr}((G \setminus e)|_J)(1) \\
&- \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in I}} P_{chr}((G/e)|_I)(k) P_{chr}((G/e)|_J)(1) - \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in J}} P_{chr}((G/e)|_I)(k) P_{chr}((G/e)|_J)(1) \\
&= \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in I}} P_{chr}((G \setminus e)|_I)(k) P_{chr}((G \setminus e)|_J)(1) + \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in J}} P_{chr}((G \setminus e)|_I)(k) P_{chr}((G \setminus e)|_J)(1) \\
&- \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in I}} P_{chr}((G/e)|_I)(k) P_{chr}((G/e)|_J)(1) - \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in J}} P_{chr}((G/e)|_I)(k) P_{chr}((G/e)|_J)(1) \\
&+ \sum_{\substack{V(G)=I \sqcup I, \\ (x, y) \in (I \times J) \cup (J \times I)}} P_{chr}((G \setminus e)|_I)(k) P_{chr}((G \setminus e)|_J)(1) \\
&= \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in I}} P_{chr}((G|_I) \setminus e)(k) P_{chr}(G|_J)(1) + \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in J}} P_{chr}(G|_I)(k) P_{chr}((G|_J) \setminus e)(1) \\
&- \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in I}} P_{chr}((G|_I)/e)(k) P_{chr}(G|_J)(1) - \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in J}} P_{chr}(G|_I)(k) P_{chr}((G|_J)/e)(1) \\
&+ \sum_{\substack{V(G)=I \sqcup I, \\ (x, y) \in (I \times J) \cup (J \times I)}} P_{chr}(G|_I)(k) P_{chr}(G|_J)(1) \\
&= \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in I}} P_{chr}(G|_I)(k) P_{chr}(G|_J)(1) + \sum_{\substack{V(G)=I \sqcup I, \\ x, y \in J}} P_{chr}(G|_I)(k) P_{chr}(G|_J)(1) \\
&+ \sum_{\substack{V(G)=I \sqcup I, \\ (x, y) \in (I \times J) \cup (J \times I)}} P_{chr}(G|_I)(k) P_{chr}(G|_J)(1) \\
&= \sum_{V(G)=I \sqcup I} P_{chr}(G|_I)(k) P_{chr}(G|_J)(1) \\
&= P_{chr}(G)(k+1).
\end{aligned}$$

So the result holds for all  $k \geq 1$ . Hence,  $P_{chr}(G) = P_{chr}(G \setminus e) - P_{chr}(G/e)$ .

Let us assume that  $G$  is connected. Note that  $G/e$  is connected. If  $e$  is a bridge, then  $G \setminus e$  is not connected; each of its connected components belongs to the augmentation ideal of  $\mathcal{H}_G$ , so their images belong to the augmentation ideal of  $\mathbb{Q}[X]$ , that is to say  $X\mathbb{Q}[X]$ ; hence,  $P_{chr}(G \setminus e) \in X^2\mathbb{Q}[X]$ , so:

$$\lambda_{chr}(G) = \frac{dP_{chr}(G)}{dX}(0) = \frac{dP_{chr}(G \setminus e)}{dX}(0) - b \frac{dP_{chr}(G/e)}{dX}(0) = 0 - \lambda_{chr}(G/e).$$

Otherwise,  $G \setminus e$  is connected, and:

$$\lambda_{chr}(G) = \frac{dP_{chr}(G)}{dX}(0) = \frac{dP_{chr}(G \setminus e)}{dX}(0) - b \frac{dP_{chr}(G/e)}{dX}(0) = \lambda_{chr}(G \setminus e) - \lambda_{chr}(G/e).$$

If  $G$  is not connected, we can write  $G = G_1 G_2$ , where  $G_1$  is connected and  $e$  is an edge of  $G_1$ . Then:

$$\begin{aligned}
\lambda_{chr}(G) &= \lambda_{chr}(G_1) \lambda_{chr}(G_2) \\
&= \begin{cases} -\lambda_{chr}(G_1/e) \lambda_{chr}(G_2) & \text{if } e \text{ is a bridge,} \\ \lambda_{chr}(G_1 \setminus e) \lambda_{chr}(G_2) - \lambda_{chr}(G_1/e) \lambda_{chr}(G_2) & \text{otherwise;} \end{cases} \\
&= \begin{cases} -\lambda_{chr}((G_1/e)G_2) & \text{if } e \text{ is a bridge,} \\ \lambda_{chr}((G_1 \setminus e)G_2) - \lambda_{chr}((G_1/e)G_2) & \text{otherwise;} \end{cases} \\
&= \begin{cases} -\lambda_{chr}(G/e) & \text{if } e \text{ is a bridge,} \\ \lambda_{chr}(G \setminus e) - \lambda_{chr}(G/e) & \text{otherwise.} \end{cases}
\end{aligned}$$

So the result holds for any graph  $G$ . □

## 2.4 Lattices attached to graphs

We here make the link with Rota's methods for proving the alternation of signs in the coefficients of chromatic polynomials.

The following order is used to prove proposition 4:

**Proposition 16** *Let  $G$  be a graph. We denote by  $\mathcal{R}(G)$  the set of equivalences  $\sim$  on  $V(G)$ , such that  $\sim \triangleleft G$ . Then  $\mathcal{R}(G)$  is partially ordered by refinement:*

$$\forall \sim, \sim' \in \mathcal{R}(G), \sim \leq \sim' \text{ if } (\forall x, y \in V(G), x \sim y \implies x \sim' y).$$

*In other words,  $\sim \leq \sim'$  if the equivalence classes of  $\sim'$  are disjoint unions of equivalence classes of  $\sim$ . Then  $(\mathcal{R}(G), \leq)$  is a bounded graded lattice. Its minimal element  $\sim_0$  is the equality; its maximal element  $\sim_1$  is the relation which equivalence classes are the connected components of  $\mathcal{R}(G)$ .*

**Proof.** Let  $\sim, \sim' \in \mathcal{R}(G)$ . We define  $\sim \wedge \sim'$  as the equivalence which classes are the connected components of the subsets  $Cl_{\sim}(x) \cap Cl_{\sim'}(y)$ ,  $x, y \in V(G)$ . By its very definition,  $\sim \wedge \sim' \triangleleft G$ , and  $\sim \wedge \sim' \leq \sim, \sim'$ . If  $\sim'' \leq \sim, \sim' \leq \sim''$  in  $\mathcal{R}(G)$ , then the equivalence classes of  $\sim$  and  $\sim'$  are disjoint union of equivalence classes of  $\sim''$ , so their intersections also are; as the equivalence classes of  $\sim''$  are connected, the connected components of these intersections are also disjoint union of equivalence classes of  $\sim''$ . This means that  $\sim'' \leq \sim \wedge \sim'$ .

We define  $\sim \vee \sim'$  as the relation defined on  $V(G)$  in the following way: for all  $x, y \in V(G)$ ,  $x \sim \vee \sim' y$  if there exists  $x_1, x'_1, \dots, x_k, x'_k \in V(G)$  such that:

$$x = x_1 \sim x'_1 \sim' x_2 \sim \dots \sim' x_k \sim' x'_k = y.$$

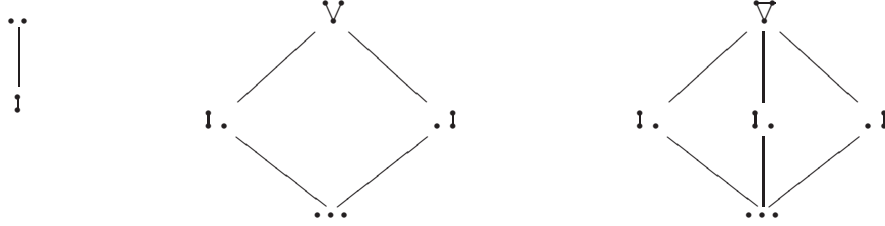
It is not difficult to prove that  $\sim \vee \sim'$  is an equivalence. Moreover, if  $x \sim y$ , then  $x \sim \vee \sim' y$  ( $x_1 = x, x'_1 = y$ ); if  $x \sim' y$ , then  $x \sim \vee \sim' y$  ( $x_1 = x'_1 = x, x_2 = x'_2 = y$ ). Let  $C$  be an equivalence class of  $\sim \vee \sim'$ , and let  $x, y \in C$ . With the preceding notations, as the equivalence classes of  $\sim$  and  $\sim'$  are connected, for all  $p \in [k]$ , there exists a path from  $x_p$  to  $x'_p$ , formed of elements  $\sim$ -equivalent, hence  $\sim \vee \sim'$ -equivalent; for all  $p \in [k-1]$ , there exists a path from  $x'_p$  to  $x'_{p+1}$ , formed of elements  $\sim'$ -equivalent, hence  $\sim \vee \sim'$ -equivalent. Concatenating these paths, we obtain a path from  $x$  to  $y$  in  $C$ , which is connected. So  $\sim \vee \sim' \in \mathcal{R}(G)$ , and  $\sim, \sim' \leq \sim \vee \sim'$ . Moreover, if  $\sim, \sim' \leq \sim''$ , then obviously  $\sim \vee \sim' \leq \sim''$ . We proved that  $\mathcal{R}(G)$  is a lattice.

For any  $\sim \in \mathcal{R}(G)$ , we put  $\deg(G) = |G| - cl(\sim)$ . Note that  $\deg(\sim_0) = 0$ . Let us assume that  $\sim$  is covered by  $\sim'$  in  $\mathcal{R}(G)$ . We denote by  $C_1, \dots, C_k$  the classes of  $\sim$ . As  $\sim \leq \sim'$ , the classes

of  $\sim'$  are disjoint unions of  $C_p$ ; as  $\sim \neq \sim'$ , one of them, denoted by  $C'$ , contains at least two  $C_p$ . As  $C'$  is connected, there is an edge in  $C'$  connecting two different  $C_p$ ; up to a reindexation, we assume that there exists an edge from  $C_1$  to  $C_2$  in  $C'$ . Then  $C_1 \sqcup C_2$  is connected, and the equivalence  $\sim''$  which classes are  $C_1 \sqcup C_2, C_3, \dots, C_k$  satisfies  $\sim \leq \sim'' \leq \sim'$ . As  $\sim'$  covers  $\sim$ ,  $\sim' = \sim''$ , so  $\deg(\sim') = |G| - k + 1 = \deg(\sim) + 1$ .  $\square$

**Remark.** This lattice is isomorphic to the one of [16]. The isomorphism between them sends a element  $\sim \in \mathcal{R}(G)$  to the partition formed by its equivalence classes.

**Examples.** We represent  $\sim \in \mathcal{R}(G)$  by  $G| \sim$ . Here are examples of  $\mathcal{R}(G)$ , represented by their Hasse graphs:



**Proposition 17** Let  $G$  be a graph. We denote by  $\mu_G$  the Möbius function of  $\mathcal{R}(G)$ .

1. If  $\sim \leq \sim'$  in  $\mathcal{R}(G)$ , then the poset  $[\sim, \sim']$  is isomorphic to  $\mathcal{R}((G| \sim') / \sim)$ .
2. For any  $\sim \leq \sim'$  in  $\mathcal{R}(G)$ ,  $\mu_G(\sim, \sim') = \lambda_{chr}((G| \sim') / \sim)$ . In particular:

$$\mu_G(\sim_0, \sim_1) = \lambda_{chr}(G).$$

**Proof.** Let  $\sim \leq \sim' \in \mathcal{R}(G)$ . If  $\sim''$  is an equivalence on  $V(G)$ , then  $\sim \leq \sim'' \leq \sim'$  if, and only if, the following conditions are satisfied:

- $\sim''$  goes to the quotient  $G / \sim$ , as an equivalence denoted by  $\overline{\sim''}$ .
- $\overline{\sim''} \in \mathcal{R}((G| \sim') / \sim)$ .

Hence, we obtain a map from  $[\sim, \sim']$  to  $\mathcal{R}((G| \sim') / \sim)$ , sending  $\sim''$  to  $\overline{\sim''}$ . It is immediate that this is a lattice isomorphism.

Let  $\sim \leq \sim' \in \mathcal{R}(G)$ . As  $[\sim, \sim']$  is isomorphic to the lattice  $\mathcal{R}((G| \sim') / \sim)$ :

$$\begin{aligned} \sum_{\sim \leq \sim'' \leq \sim'} \lambda_{chr}((G| \sim'') / \sim) &= \sum_{\overline{\sim''} \in \mathcal{R}((G| \sim') / \sim)} \lambda_{chr}(((G / \sim') / \sim) | \overline{\sim''}) \\ &= P_{chr}((G| \sim') / \sim)(1) \\ &= \begin{cases} 1 & \text{if } (G| \sim') / \sim \text{ is totally disconnected,} \\ 0 & \text{otherwise;} \end{cases} \\ &= \begin{cases} 1 & \text{if } \sim = \sim', \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence,  $\mu_G(\sim, \sim') = \lambda_{chr}((G| \sim') / \sim)$ .  $\square$

**Remark.** We now use the notion of incidence algebra of a family of posets exposed in [17]. We consider the family of posets:

$$\{[\sim, \sim'] \mid G \in \mathcal{G}, \sim \leq \sim' \text{ in } \mathcal{R}(G)\}.$$

It is obviously interval closed. We define an equivalence relation on this family as the one generated by  $[\sim, \sim'] \equiv \mathcal{R}((G| \sim') / \sim)$ . The incidence bialgebra associated to this family is  $(\mathcal{H}_{\mathcal{G}}, m, \delta)$ .

**Proposition 18** *Let  $G$  a graph.*

1. *Let  $G_1, \dots, G_k$  be the connected components of  $G$ . Then  $\mathcal{R}(G) \approx \mathcal{R}(G_1) \times \dots \times \mathcal{R}(G_k)$ .*
2. *Let  $e$  be a bridge of  $G$ . Then  $\mathcal{R}(G) \approx \mathcal{R}(G/e) \times \mathcal{R}(\mathbf{1})$ .*
3. *We consider the following map:*

$$\zeta_G : \begin{cases} \mathcal{R}(G) & \longrightarrow & \mathcal{P}(E(G)) \\ \sim & \longrightarrow & E(G|\sim). \end{cases}$$

*This map is injective; for any  $\sim, \sim' \in \mathcal{R}(G)$ ,  $\sim \leq \sim'$  if, and only if,  $\zeta_G(\sim) \subseteq \zeta_G(\sim')$ . Moreover,  $\zeta_G$  is bijective if, and only if,  $G$  is a forest.*

**Proof.** 1. If  $G, H$  are graphs and  $\sim$  is an equivalence on  $V(GH)$ , then  $\sim \triangleleft GH$  if, and only if:

- $\sim|_{V(G)} \triangleleft G$ .
- $\sim|_{V(H)} \triangleleft H$ .
- For any  $x, y \in V(G) \sqcup V(H)$ ,  $(x \sim y) \implies ((x, y) \in V(G)^2 \sqcup V(H)^2)$ .

Hence, the map sending  $\sim$  to  $(\sim|_{V(G)}, \sim|_{V(H)})$  from  $\mathcal{R}(GH)$  to  $\mathcal{R}(G) \times \mathcal{R}(H)$  is an isomorphism; the first point follows.

2. Note that  $\mathcal{R}(\mathbf{1}) = \{\bullet, \mathbf{1}\}$ , with  $\bullet \leq \mathbf{1}$ . By the first point, it is enough to prove it if  $G$  is connected. Let us put  $e = \{x', x''\}$ ,  $G'$ , respectively  $G''$ , the connected components of  $G \setminus e$  containing  $x'$ , respectively  $x''$ . We define a map  $\psi : \mathcal{R}(G/e) \times \mathcal{R}(\mathbf{1})$  to  $\mathcal{R}(G)$  in the following way: if  $\preceq \triangleleft \mathcal{R}(G/e)$ ,

- $\psi(\preceq, \mathbf{1}) = \sim$ , defined by  $x \sim y$  if  $\overline{x} \preceq \overline{y}$ . This is clearly an equivalence; moreover,  $x' \sim x''$ . if  $x \sim y$ , there exists a path from  $\overline{x}$  to  $\overline{y}$  in  $G/e$ , formed by vertices  $\preceq$ -equivalent to  $\overline{x}$  and  $\overline{y}$ . Adding edges  $e$  if needed in this path, we obtain a path from  $x$  to  $y$  in  $G$ , formed by vertices  $\sim$ -equivalent to  $x$  and  $y$ ; hence,  $\sim \triangleleft G$ .
- $\psi(\preceq, \bullet) = \sim$ , defined by  $x \sim y$  if  $\overline{x} \preceq \overline{y}$  and  $(x, y) \in V(G')^2 \sqcup V(G'')^2$ . This is clearly an equivalence; moreover, we do not have  $x' \sim x''$ . If  $x \sim y$ , let us assume for example that both of them belong to  $G'$ . There is a path in  $G \setminus e$  from  $\overline{x}$  to  $\overline{y}$ , formed by vertices formed by vertices  $\preceq$ -equivalent to  $\overline{x}$  and  $\overline{y}$ . We choose such a path of minimal length. If this path contains vertices belonging to  $G''$ , as  $e$  is a bridge of  $G$ , it has the form:

$$\overline{x} - \dots - \overline{x'} - \dots - \overline{x'} - \dots - \overline{y}.$$

Hence, we can obtain a shorter path from  $\overline{x}$  to  $\overline{y}$ : this is a contradiction. So all the vertices of this path belong to  $G'$ ; hence, they are all  $\sim$ -equivalent. Finally,  $\sim \triangleleft G$ .

Let us assume that  $\psi(\preceq, \mathbf{1}) = \psi(\preceq', \mathbf{1}) = \sim$ . If  $\overline{x} \preceq \overline{y}$ , then  $x \sim y$ , so  $\overline{x} \preceq \overline{y}$ ; by symmetry,  $\preceq = \preceq'$ . Let us assume that  $\psi(\preceq, \bullet) = \psi(\preceq', \bullet) = \sim$ . If  $\overline{x} \preceq \overline{y}$ :

- If  $x, y \in V(G')$  or  $x, y \in V(G'')$ , then  $x \sim y$ , so  $\overline{x} \preceq \overline{y}$ .
- If  $(x, y) \in V(G') \times V(G'')$  or  $(x, y) \in V(G'') \times V(G')$ , up to a permutation we can assume that  $x \in V(G')$  and  $y \in V(G'')$ . As  $\preceq \triangleleft G/e$ , there exists a path from  $\overline{x}$  to  $\overline{y}$  formed by  $\preceq$ -equivalent vertices. This path necessarily goes via  $\overline{x'} = \overline{x''}$ . Hence,  $x \sim x'$  and  $y \sim x''$ , so  $\overline{x} \preceq \overline{x'}$  and  $\overline{y} \preceq \overline{x'}$ , and finally  $\overline{x} \preceq \overline{y}$ .



By symmetry,  $\overline{\sim} = \overline{\sim}'$ . We proved that  $\psi$  is injective.

Let  $\sim \triangleleft G$ . If  $x' \sim x''$ , then  $\sim$  goes through the quotient  $G/e$  and gives an equivalence  $\overline{\sim} \triangleleft G/e$ . Moreover,  $\psi(\overline{\sim}, \mathbf{1}) = \sim$ . Otherwise,  $\sim \triangleleft G \setminus e = G'G''$ ; let us denote the equivalence classes of  $\sim$  by  $C_1, \dots, C_{k+l}$ , with  $x' \in C_1$ ,  $x'' \in C_{k+1}$ ,  $C_1, \dots, C_k \subseteq V(G')$ ,  $C_{k+1}, \dots, C_{k+l} \subseteq V(G'')$ . Let  $\overline{\sim}$  the equivalence on  $V(G/e)$  which equivalence classes are  $\overline{C_1} \sqcup \overline{C_{k+1}}, \overline{C_2}, \dots, \overline{C_k}, \overline{C_{k+2}}, \dots, \overline{C_{k+l}}$ . Then  $\overline{\sim} \triangleleft G/e$  and  $\psi(\overline{\sim}, \bullet) = \sim$ . We proved that  $\psi$  is surjective.

It is immediate that  $\psi(\overline{\sim}_1, \sim_2) \leq \psi(\overline{\sim}'_1, \sim'_2)$  if, and only if,  $\overline{\sim}_1 \leq \overline{\sim}'_1$  and  $\sim_2 \leq \sim'_2$ . So  $\psi$  is a lattice isomorphism.

3. Let  $\sim, \sim'$  be elements of  $\mathcal{R}(G)$ . If  $\sim \leq \sim'$ , then the connected components of  $G| \sim'$  are disjoint unions of connected components of  $G| \sim$ , so  $E(G| \sim) \subseteq E(G| \sim')$ .

If  $E(G| \sim) \subseteq E(G| \sim')$ , then the connected components of  $G| \sim'$  are disjoint unions of connected components of  $G| \sim$ , so  $\sim \leq \sim'$ .

Consequently, if  $\zeta_G(\sim) = \zeta_G(\sim')$ , then  $\sim \leq \sim'$  and  $\sim' \leq \sim$ , so  $\sim = \sim'$ :  $\zeta_G$  is injective.

Let us assume that  $\zeta_G$  is surjective. Let  $e \in E(G)$ ; we consider  $\sim \in \mathcal{R}(G)$ , such that  $\zeta_G(\sim) = E(G) \setminus e$ . In other words,  $G| \sim = G \setminus e$ . Hence,  $\sim \neq \sim_1$ , so  $cl(\sim) < cl(\sim_1)$ :  $G| \sim$  has strictly more connected components than  $G$ . This proves that  $e$  is a bridge, so  $G$  is a forest.

Let us assume that  $G$  is a forest. We denote by  $k$  the number of its edges. As any edge of  $G$  is a bridge, by the second point,  $\mathcal{R}(G)$  is isomorphic to  $\mathcal{R}(\mathbf{1})^k \times \mathcal{R}(\cdot)^{cc(G)}$ , so is of cardinal  $2^k \times 1^{cc(G)} = 2^k$ . Hence,  $\zeta_G$  is surjective.  $\square$

**Remark.** As a consequence, isomorphic posets may be associated to non-isomorphic graphs: for an example, take two non-isomorphic trees with the same degree.

## 2.5 Applications

**Corollary 19** *Let  $G$  be a graph. We put  $P_{chr}(G) = a_0 + \dots + a_n X^n$ .*

1.  $\lambda_{chr}(G)$  is non-zero, of sign  $(-1)^{deg(G)}$ .
2.
  - For any  $i$ ,  $a_i \neq 0$  if, and only if,  $cc(G) \leq i \leq |G|$ .
  - If  $cc(G) \leq i \leq |G|$ , the sign of  $a_i$  is  $(-1)^{|G|-i}$ .
3.  $-a_{|G|-1}$  is the number of edges of  $|G|$ .

**Proof.** 1. For any graph  $G$ , we put  $\tilde{\lambda}_{chr}(G) = (-1)^{deg(G)} \lambda_{chr}(G)$ . This defines an element  $\tilde{\lambda} \in M_G$ . Let us prove that for any edge  $e$  of  $G$ :

$$\tilde{\lambda}_{chr}(G) = \begin{cases} \tilde{\lambda}_{chr}(G/e) & \text{if } e \text{ is a bridge,} \\ \tilde{\lambda}_{chr}(G \setminus e) + \tilde{\lambda}_{chr}(G/e) & \text{otherwise.} \end{cases}$$

We proceed by induction on the number  $k$  of edges of  $G$ . If  $k = 0$ , there is nothing to prove. Let us assume the result at all ranks  $< k$ , with  $k \geq 1$ . Let  $e$  be an edge of  $G$ . We shall apply the induction hypothesis to  $G/e$  and  $G \setminus e$ . Note that  $cc(G/e) = cc(G)$  and  $|G/e| = |G| - 1$ , so  $deg(G/e) = deg(G) - 1$ .

- If  $e$  is a bridge, then:

$$\lambda_{chr}(G) = -(-1)^{deg(G/e)} \tilde{\lambda}_{chr}(G/e) = (-1)^{deg(G)} \tilde{\lambda}_{chr}(G/e).$$

- If  $e$  is not a bridge, then  $cc(G \setminus e) = cc(G)$ , and  $|G \setminus e| = |G|$ , so  $deg(G \setminus e) = deg(G)$ . Hence:

$$\begin{aligned}\lambda_{chr}(G/e) &= (-1)^{deg(G \setminus e)} \tilde{\lambda}_{chr}(G \setminus e) - (-1)^{deg(G/e)} \tilde{\lambda}_{chr}(G/e) \\ &= (-1)^{deg(G)} \tilde{\lambda}_{chr}(G \setminus e) + (-1)^{deg(G)} \tilde{\lambda}_{chr}(G/e) \\ &= (-1)^{deg(G)} (\tilde{\lambda}_{chr}(G \setminus e) + \tilde{\lambda}_{chr}(G/e)).\end{aligned}$$

So the result holds for all graph  $G$ .

If  $G$  has no edge, then  $deg(G) = 0$  and  $\lambda_{chr}(G) = \tilde{\lambda}_{chr}(G) = 1$ . An easy induction on the number of edges proves that for any graph  $G$ ,  $\tilde{\lambda}_{chr}(G) \geq 1$ .

2. By corollary 12, for any  $i$ :

$$a_i = \sum_{\sim \triangleleft G, cl(\sim)=i} \lambda_{chr}(G| \sim) = \sum_{\sim \triangleleft G, cl(\sim)=i} (-1)^{|G|-i} \tilde{\lambda}_{chr}(G| \sim) = (-1)^{|G|-i} \sum_{\sim \triangleleft G, cl(\sim)=i} \tilde{\lambda}_{chr}(G| \sim).$$

As for any graph  $H$ ,  $\tilde{\lambda}_{chr}(H) > 0$ , this is non-zero if, and only if, there exists a relation  $\sim \triangleleft G$ , such that  $cl(\sim) = i$ . If this holds, the sign of  $a_i$  is  $(-1)^{|G|-i}$ . It remains to prove that there exists a relation  $\sim \triangleleft G$ , such that  $cl(\sim) = i$  if, and only if,  $cc(G) \leq i \leq |G|$ .

$\implies$ . If  $\sim \triangleleft G$ , with  $cl(\sim) = i$ , as the equivalence classes of  $\sim$  are connected, each connected component of  $G$  is a union of classes of  $\sim$ , so  $i \geq cc(G)$ . Obviously,  $i \leq |G|$ .

$\impliedby$ . We proceed by decreasing induction on  $i$ . If  $i = |G|$ , then the equality of  $V(G)$  answers the question. Let us assume that  $cc(G) \leq i < |G|$  and that the result holds at rank  $i + 1$ . Let  $\sim' \triangleleft G$ , with  $cl(\sim') = i + 1$ . We denote by  $I_1, \dots, I_{i+1}$  the equivalence classes of  $\sim'$ . As  $I_1, \dots, I_{i+1}$  are connected, the connected components of  $G$  are union of  $I_p$ ; as  $i + 1 > cc(G)$ , one of the connected components of  $G$ , which we call  $G'$ , contains at least two equivalence classes of  $\sim'$ . As  $G'$  is connected, there exists an edge in  $G'$ , relation two vertices into different equivalence classes of  $\sim'$ ; up to a reindexation, we assume that they are  $I_1$  and  $I_2$ . Hence,  $I_1 \sqcup I_2$  is connected. We consider the relation  $\sim$  which equivalence classes are  $I_1 \sqcup I_2, I_3, \dots, I_{i+1}$ : then  $\sim \triangleleft G$  and  $cl(\sim) = i$ .

3. For  $i = |G| - 1$ , we have to consider relations  $\sim \triangleleft G$  such that  $cl(\sim) = |G| - 1$ . These equivalences are in bijection with edges, via the map  $\zeta_G$  of proposition 18. For such an equivalence,  $G| \sim = \mathbf{1} \cdot |G|^{-1}$ , so  $\lambda_{chr}(G| \sim) = -1$ . Finally,  $a_i = -|E(V)|$ .  $\square$

**Remark.** The result on the signs of the coefficients of  $P_{chr}(G)$  is due to Rota [16], who proved it using the Möbius function of the poset of proposition 18.

**Corollary 20** *Let  $G$  be a graph;  $|\lambda_{chr}(G)| = 1$  if, and only if,  $G$  is a forest, that is to say that any edge of  $G$  is a bridge.*

**Proof.**  $\impliedby$ . We proceed by induction on the number of edges  $k$  of  $G$ . If  $k = 0$ ,  $\lambda_{chr}(G) = 1$ . If  $k \geq 1$ , let us choose an edge of  $G$ ; it is a bridge and  $G/e$  is also a forest, so  $|\lambda_{chr}(G)| = |\lambda_{chr}(G/e)| = 1$ .

$\implies$ . If  $G$  is not a forest, there exists an edge  $e$  of  $G$  which is not a bridge. Then:

$$|\lambda_{chr}(G)| = |\lambda_{chr}(G \setminus e)| + |\lambda_{chr}(G/e)| \geq 1 + 1 = 2.$$

So  $|\lambda_{chr}(G)| \neq 1$ .  $\square$

**Lemma 21** *If  $G$  is a graph and  $e$  is a bridge of  $G$ , then:*

$$\lambda_{chr}(G) = -\lambda_{chr}(G \setminus e) = -\lambda_{chr}(G/e).$$

**Proof.** We already proved in proposition 18 that  $\lambda_{chr}(G) = -\lambda_{chr}(G/e)$ . Let us prove that  $\lambda_{chr}(G) = \lambda_{chr}(G \setminus e)$  by induction on the number  $k$  of edges of  $G$  which are not bridges. If  $k = 0$ , then  $G$  and  $G \setminus e$  are forests with  $n$  vertices,  $cc(G \setminus e) = cc(G) + 1$  and:

$$\lambda_{chr}(G) = -\lambda_{chr}(G \setminus e) = (-1)^{deg(G)}.$$

Let us assume the result at rank  $k - 1$ ,  $k \geq 1$ . Let  $f$  be an edge of  $G$  which is not a bridge of  $G$ .

$$\begin{aligned} \lambda_{chr}(G) &= \lambda_{chr}(G \setminus f) - \lambda_{chr}(G/f) \\ &= -\lambda_{chr}((G \setminus f) \setminus e) + \lambda_{chr}((G/f) \setminus e) \\ &= -\lambda_{chr}((G \setminus e) \setminus f) + \lambda_{chr}((G \setminus e)/f) \\ &= -\lambda_{chr}(G \setminus e). \end{aligned}$$

So the result holds for any bridge of any graph.  $\square$

**Proposition 22** 1. Let  $G$  and  $H$  be two graphs, with  $V(G) = V(H)$  and  $E(G) \subseteq E(H)$ . Then:

$$|\lambda_{chr}(G)| \leq |\lambda_{chr}(H)| + cc(G) - cc(H) - \sharp(E(H) - E(G)) \leq |\lambda_{chr}(H)|.$$

Moreover, if  $cc(G) = cc(H)$ , then  $|\lambda_{chr}(G)| = |\lambda_{chr}(H)|$  if, and only if,  $G = H$ .

2. For any graph  $G$ ,  $|\lambda_{chr}(G)| \leq (|G| - 1)!$ , with equality if, and only if,  $G$  is complete.

**Proof.** 1. We put  $k = \sharp(E(H) \setminus E(G))$ . There exists a sequence  $e_1, \dots, e_k$  of edges of  $H$  such that:

$$G_0 = G, \quad G_k = H, \quad \forall i \in [k], G_{i-1} = G_i \setminus e_i.$$

For all  $i$ ,  $cc(G_i) = cc(G_{i-1}) + 1$  if  $e_i$  is a bridge of  $G_i$ , and  $cc(G_i) = cc(G_{i-1})$  otherwise. Hence,  $cc(G) - cc(H) \leq k$ . We denote by  $I$  the set of indices  $i$  such that  $cc(G_i) = cc(G_{i-1})$ ; then  $\sharp I = k - cc(G) + cc(H)$ . Moreover:

$$|\lambda_{chr}(G_i)| = \begin{cases} |\lambda_{chr}(G_{i-1})| + |\lambda_{chr}((G_i)/e_i)| > |\lambda_{chr}(G_{i-1})| & \text{if } i \in I, \\ |\lambda_{chr}(G_{i-1})| & \text{if } i \notin I. \end{cases}$$

As a conclusion,  $|\lambda_{chr}(G)| \leq |\lambda_{chr}(H)| - \sharp I = |\lambda_{chr}(H)| + cc(G) - cc(H) - k \leq |\lambda_{chr}(H)|$ .

If  $cc(G) = cc(H)$  and  $|\lambda_{chr}(G)| = |\lambda_{chr}(H)|$ , then  $k = 0$ , so  $G = H$ .

2. We put  $n = |G|$ . We apply the first point with  $H$  the complete graph such that  $V(H) = V(G)$ . We already observed that  $|\lambda_{chr}(H)| = (n - 1)!$ , so:

$$|\lambda_{chr}(G)| \leq (n - 1)!.$$

If  $G$  is not connected, there exist graphs  $G_1, G_2$  such that  $G = G_1 G_2$ ,  $n_1 = |G_1| < n$ ,  $n_2 = |G_2| < n$ . Hence:

$$|\lambda_{chr}(G)| = |\lambda_{chr}(G_1)| |\lambda_{chr}(G_2)| \leq (n_1 - 1)! (n_2 - 1)! \leq (n_1 + n_2 - 2) < (n - 1)!.$$

If  $G$  is connected, then  $cc(G) = cc(H)$ : if  $|\lambda_{chr}(G)| = |\lambda_{chr}(H)|$ , then  $G = H$ .  $\square$

## 2.6 Values at negative integers

**Theorem 23** *Let  $k \geq 1$  and  $G$  a graph. Then  $(-1)^{|G|}P_{chr}(G)(-k)$  is the number of families  $((I_1, \dots, I_k), O_1, \dots, O_k)$  such that:*

- $I_1 \sqcup \dots \sqcup I_k = V(G)$  (note that one may have empty  $I_p$ 's).
- For all  $1 \leq i \leq k$ ,  $O_i$  is an acyclic orientation of  $G|_{I_i}$ .

In particular,  $(-1)^{|P|}P_{chr}(G)(-1)$  is the number of acyclic orientations of  $G$ .

**Proof.** Note that for any graph  $G$ ,  $(-1)^{|G|}P_{chr}(G)(-X) = P_{-1}(G)(X)$ . Moreover:

- If  $G$  is totally disconnected,  $P_{-1}(G)(1) = 1$ .
- If  $G$  has an edge  $e$ ,  $P_{-1}(G)(1) = P_{-1}(G \setminus e)(1) + P_{-1}(G/e)$ .

An induction on the number of edges of  $G$  proves that  $P_{-1}(G)(1)$  is indeed the number of acyclic orientations of  $G$ . If  $k \geq 2$ :

$$\begin{aligned} P_{-1}(G)(k) &= P_{-1}(G)(1 + \dots + 1) \\ &= \Delta^{(k-1)} \circ P_{-1}(G)(1, \dots, 1) \\ &= P_{-1}^{\otimes k} \circ \Delta^{(k-1)}(G)(1, \dots, 1) \\ &= \sum_{V(G)=I_1 \sqcup \dots \sqcup I_k} P_{-1}(G|_{I_1})(1) \dots P_{-1}(G|_{I_k})(1). \end{aligned}$$

The case  $k = 1$  implies the result. □

We recover the interpretation of Stanley [18]:

**Corollary 24** *Let  $k \geq 1$  and  $G$  a graph. Then  $(-1)^{|G|}P_{chr}(G)(-k)$  is the number of pairs  $(f, O)$  where*

- $f$  is a map from  $V(G)$  to  $[k]$ .
- $O$  is an acyclic orientation of  $G$ .
- If there is an oriented edge from  $x$  to  $y$  in  $V(G)$  for the orientation  $O$ , then  $f(x) \leq f(y)$ .

**Proof.** Let  $A$  be the set of families defined in theorem 23, and  $B$  be the set of pairs defined in corollary 24. We define a bijection  $\theta : A \rightarrow B$  in the following way: if  $((I_1, \dots, I_k), O_1, \dots, O_k) \in A$ , we put  $\theta((I_1, \dots, I_k), O_1, \dots, O_k) = (f, O)$ , such that:

1.  $f^{-1}(p) = I_p$  for any  $p \in [k]$ .
2. If  $e = \{x, y\} \in E(G)$ , we put  $f(x) = i$  and  $f(y) = j$ . If  $i = j$ , then  $e$  is oriented as in  $O_i$ . Otherwise, if  $i < j$ ,  $e$  is oriented from  $i$  to  $j$  if  $i < j$  and from  $j$  to  $i$  if  $i > j$ .

Note that  $O$  is indeed acyclic: if there is an oriented path from  $x$  to  $y$  in  $G$  of length  $\geq 1$ , then  $f$  increases along this path. If  $f$  remains constant, as  $O_{f(x)}$  is acyclic,  $x \neq y$ . Otherwise,  $f(x) < f(y)$ , so  $x \neq y$ . It is then not difficult to see that  $\theta$  is bijective. □

### 3 Non-commutative version

#### 3.1 Non-commutative Hopf algebra of graphs

**Definition 25** 1. An indexed graph is a graph  $G$  such that  $V(G) = [n]$ , with  $n \geq 0$ . The set of indexed graphs is denoted by  $\mathcal{G}$ .

2. Let  $G = ([n], E(G))$  be an indexed graph and let  $I \subseteq [n]$ . There exists a unique increasing bijection  $f : I \rightarrow [k]$ , where  $k = \sharp I$ . We denote by  $G|_I$  the indexed graph defined by:

$$G|_I = ([k], \{\{f(x), f(y)\} \mid \{x, y\} \in E(G), x, y \in I\}).$$

3. Let  $G$  be an indexed graph and  $\sim \triangleleft G$ .

(a) The graph  $G|_{\sim}$  is an indexed graph.

(b) We order the elements of  $V(G)/\sim$  by their minimal elements; using the unique increasing bijection from  $V(G)/\sim$  to  $[k]$ ,  $G/\sim$  becomes an indexed graph.

4. Let  $G = ([k], E(G))$  and  $H = ([l], E(H))$  be indexed graphs. The indexed graph  $GH$  is defined by:

$$\begin{aligned} V(GH) &= [k + l], \\ E(GH) &= E(G) \sqcup \{\{x + k, y + l\} \mid \{x, y\} \in E(H)\}. \end{aligned}$$

The Hopf algebra  $(\mathcal{H}_{\mathcal{G}}, m, \Delta)$  is, as its commutative version, introduced in [17]:

**Theorem 26** 1. We denote by  $\mathcal{H}_{\mathcal{G}}$  the vector space generated by indexed graphs. We define a product  $m$  and two coproducts  $\Delta$  and  $\delta$  on  $\mathcal{H}_{\mathcal{G}}$  in the following way:

$$\begin{aligned} \forall G, H \in \mathcal{G}, m(G \otimes H) &= GH, \\ \forall G = ([n], E(G)) \in \mathcal{G}, \Delta(G) &= \sum_{I \subseteq [n]} G|_I \otimes G|_{[n] \setminus I}, \\ \forall G \in \mathcal{G}, \delta(G) &= \sum_{\sim \triangleleft G} G/\sim \otimes G|_{\sim}. \end{aligned}$$

Then  $(\mathcal{H}_{\mathcal{G}}, m, \Delta)$  is a graded cocommutative Hopf algebra, and  $(\mathcal{H}_{\mathcal{G}}, m, \delta)$  is a bialgebra.

2. Let  $\varpi : \mathcal{H}_{\mathcal{G}} \rightarrow \mathcal{H}_{\mathcal{G}}$  be the surjection sending an indexed graph to its isoclass.

(a)  $\varpi : (\mathcal{H}_{\mathcal{G}}, m, \Delta) \rightarrow (\mathcal{H}_{\mathcal{G}}, m, \Delta)$  is a surjective Hopf algebra morphism.

(b)  $\varpi : (\mathcal{H}_{\mathcal{G}}, m, \delta) \rightarrow (\mathcal{H}_{\mathcal{G}}, m, \delta)$  is a surjective bialgebra morphism.

(c) We put  $\rho = (Id \otimes \varpi) \circ \delta : \mathcal{H}_{\mathcal{G}} \rightarrow \mathcal{H}_{\mathcal{G}} \otimes \mathcal{H}_{\mathcal{G}}$ . This defines a coaction of  $(\mathcal{H}_{\mathcal{G}}, m, \delta)$  on  $\mathcal{H}_{\mathcal{G}}$ ; moreover,  $(\mathcal{H}_{\mathcal{G}}, m, \Delta)$  is a Hopf algebra in the category of  $(\mathcal{H}_{\mathcal{G}}, m, \delta)$ -comodules.

**Proof.** 1. Similar as the proof of propositions 2 and 4.

2. Points (a) and (b) are immediate; point (c) is proved in the same way as theorem 7.  $\square$

**Examples.**

$$\begin{aligned}
\Delta(\cdot_1) &= \cdot_1 \otimes 1 + 1 \otimes \cdot_1, \\
\Delta(\mathbf{1}_1^2) &= \mathbf{1}_1^2 \otimes 1 + 1 \otimes \mathbf{1}_1^2 + \cdot_1 \otimes \cdot_1, \\
\Delta({}^2\nabla_1^3) &= {}^2\nabla_1^3 \otimes 1 + 1 \otimes {}^2\nabla_1^3 + 3\cdot_1 \otimes \mathbf{1}_1^2 + 3\mathbf{1}_1^2 \otimes \cdot_1, \\
\Delta({}^2\mathbf{V}_1^3) &= {}^2\mathbf{V}_1^3 \otimes 1 + 1 \otimes {}^2\mathbf{V}_1^3 + 2\mathbf{1}_1^2 \otimes \cdot_1 + \cdot_1 \cdot_2 \otimes \cdot_1 + 2\cdot_1 \otimes \mathbf{1}_1^2 + \cdot_1 \otimes \cdot_1 \cdot_2;
\end{aligned}$$

$$\begin{aligned}
\delta(\cdot_1) &= \cdot_1 \otimes \cdot_1, \\
\delta(\mathbf{1}_1^2) &= \cdot_1 \otimes \mathbf{1}_1^2 + \mathbf{1}_1^2 \otimes \cdot_1 \cdot_2, \\
\delta({}^2\nabla_1^3) &= \cdot_1 \otimes {}^2\nabla_1^3 + \mathbf{1}_1^2 \otimes (\cdot_1 \mathbf{1}_2^3 + \cdot_2 \mathbf{1}_1^3 + \cdot_3 \mathbf{1}_1^2) + {}^2\nabla_1^3 \otimes \cdot_1 \cdot_2 \cdot_3, \\
\delta({}^2\mathbf{V}_1^3) &= \cdot_1 \otimes {}^2\mathbf{V}_1^3 + \mathbf{1}_1^2 \otimes (\cdot_1 \mathbf{1}_2^3 + \cdot_3 \mathbf{1}_1^2) + {}^2\mathbf{V}_1^3 \otimes \cdot_1 \cdot_2 \cdot_3.
\end{aligned}$$

**Remark.**  $(\mathcal{H}\mathcal{G}, m, \Delta)$  is not a bialgebra in the category of  $(\mathcal{H}\mathcal{G}, m, \delta)$ -comodules, as shown in the following example:

$$\begin{aligned}
(\Delta \otimes Id) \circ \delta({}^2\mathbf{V}_1^3) &= \Delta(\cdot_1) \otimes {}^2\mathbf{V}_1^3 + \Delta(\mathbf{1}_1^2) \otimes (\cdot_2 \mathbf{1}_1^3 + \mathbf{1}_1^2 \cdot_3) + \Delta({}^2\nabla_1^3) \otimes \cdot_1 \cdot_2 \cdot_3, \\
m_{2,4}^3 \circ (\delta \otimes \delta) \circ \Delta({}^2\mathbf{V}_1^3) &= \Delta(\cdot_1) \otimes {}^2\mathbf{V}_1^3 + \Delta(\mathbf{1}_1^2) \otimes (\cdot_1 \mathbf{1}_1^2 + \mathbf{1}_1^2 \cdot_3) + \Delta({}^2\nabla_1^3) \otimes \cdot_1 \cdot_2 \cdot_3.
\end{aligned}$$

### 3.2 Reminders on WQSym

Let us recall the construction of **WQSym** [14].

**Definition 27** 1. Let  $w$  be a word in  $\mathbb{N} \setminus \{0\}$ . We shall say that  $w$  is packed if:

$$\forall i, j \geq 0, (i \leq j \text{ and } j \text{ appears in } w) \implies (i \text{ appears in } w).$$

2. Let  $w = x_1 \dots x_k$  a word in  $\mathbb{N}$ . There exists a unique increasing bijection  $f$  from  $\{x_1, \dots, x_k\}$  to  $[l]$ , with  $l \geq 0$ ; the packed word  $\text{Pack}(w)$  is  $f(x_1) \dots f(x_k)$ .
3.  $w = x_1 \dots x_k$  a word in  $\mathbb{N} \setminus \{0\}$  and  $I \subseteq \mathbb{N} \setminus \{0\}$ . The word  $w|_I$  is the word obtained by taking the letters of  $w$  which are in  $I$ .

The Hopf algebra **WQSym** has the set of packed words for basis. If  $w = w_1 \dots w_k$  and  $w' = w'_1 \dots w'_l$  are packed words, then:

$$w.w' = \sum_{\substack{w''=w''_1 \dots w''_{k+l}, \\ \text{Pack}(w''_1 \dots w''_k)=w, \\ \text{Pack}(w''_{k+1} \dots w''_{k+l})=w'}} w''.$$

For any packed word  $w$ :

$$\Delta(P_w) = \sum_{i=0}^{\max(w)} w|_I \otimes \text{Pack}(w_{[\max(w)] \setminus I}).$$

This Hopf algebra has a polynomial representation: we fix a infinite totally ordered alphabet  $X$ ; the set of words in  $X$  is denoted by  $X^*$ . For any packed word  $w$ , we consider the element:

$$P_w(X) = \sum_{w' \in X^*, \text{Pack}(w')=w} w' \in \mathbb{Q}\langle X \rangle.$$

Then for any packed words  $w, w'$ :

$$P_w(X)P_{w'}(X) = P_{w.w'}(X).$$

If  $X$  and  $Y$  are two totally ordered alphabets,  $X \sqcup Y$  is also totally ordered: for all  $x \in X$ ,  $y \in Y$ ,  $x \leq y$ . Projecting from  $\mathbb{Q}\langle\langle X \sqcup Y \rangle\rangle$  to  $\mathbb{Q}\langle\langle X \rangle\rangle \otimes \mathbb{Q}\langle\langle Y \rangle\rangle$ , for any packed word  $w$ , with Sweedler's notation  $\Delta(w) = \sum w^{(1)} \otimes w^{(2)}$ :

$$P_w(X \sqcup Y) = \sum P_{w^{(1)}}(X) \otimes P_{w^{(2)}}(Y).$$

### 3.3 Non-commutative chromatic polynomial

**Notations.** A set partition is a partition of a set  $[n]$ , with  $n \geq 0$ . The set of set partitions is denoted by  $\mathcal{SP}$ .

**Theorem 28** 1. For any packed word  $w$  of length  $n$  and of maximal  $k$ , we denote by  $p(w)$  the set partition  $\{w^{-1}(1), \dots, w^{-1}(k)\}$ . For any set partition  $\varpi \in \mathcal{SP}$ , we put:

$$W_\varpi = \sum_{w \in \mathcal{PW}, p(w)=\varpi} w.$$

These elements are a basis of a cocommutative Hopf subalgebra of  $\mathbf{WQSym}$ , denoted by  $\mathbf{WSym}$ .

2. The following map is a Hopf algebra morphism:

$$\mathbf{P}_{chr} : \begin{cases} \mathcal{H}_{\mathcal{G}} & \longrightarrow \mathbf{WQSym} \\ G \in \mathcal{G} & \longrightarrow \sum_{f \in \mathbb{PVC}(G)} f(1) \dots f(|G|). \end{cases}$$

The image of  $\mathbf{P}_{chr}$  is  $\mathbf{WSym}$ . For any indexed graph  $G$ :

$$\mathbf{P}_{chr}(G) = \sum_{\varpi \in \mathbb{IP}(G)} W_\varpi.$$

**Proof.** 2. Let us prove that  $\mathbf{P}_{chr}$  is a Hopf algebra morphism. We shall use the polynomial representation of  $\mathbf{WQSym}$ . Choosing an infinite totally ordered alphabet  $X$ , for any indexed graph  $G$ :

$$\mathbf{P}_{chr}(G)(X) = \sum_{w \in \mathbb{PVC}(G)} P_w(X) = \sum_{f \in \mathbb{VC}(G, X)} f(1) \dots f(n).$$

Seen in this way, as a noncommutative formal series,  $\mathbf{P}_{chr}(G)(X)$  is the chromatic symmetric function in noncommuting variables introduced in [9]; see also [15]. If  $G$  and  $H$  are graphs on respectively  $[k]$  and  $[l]$ , denoting that any valid coloring of  $GH$  is obtained by concatenating a valid coloring of  $G$  and a valid coloring of  $H$ :

$$\mathbf{P}_{chr}(GH)(X) = \sum_{f \in \mathbb{VC}(G, X), g \in \mathbb{VC}(H, X)} f(1) \dots f(k)g(1) \dots g(l) = \mathbf{P}_{chr}(G)(X)\mathbf{P}_{chr}(H)(X).$$

If  $X$  and  $Y$  are two ordered alphabets, for any graph  $G$  on  $[n]$ :

$$\begin{aligned} \mathbf{P}_{chr}(G)(X \sqcup Y) &= \sum_{\substack{I \subseteq [n], \\ f \in \mathbb{VC}(G|_I, X), \\ g \in \mathbb{VC}(G|_{[n] \setminus I}, Y)}} \prod_{i \in I} f(i) \otimes \prod_{i \notin I} g(i) \\ &= \sum_{I \subseteq [n]} \mathbf{P}_{chr}(G|_I)(X) \otimes \mathbf{P}_{chr}(G|_{[n] \setminus I})(Y). \end{aligned}$$

So  $\mathbf{P}_{chr}$  is a Hopf algebra morphism.

The formula for  $\mathbf{P}_{chr}(G)$  is immediate. This implies that  $\mathbf{P}_{chr}(\mathcal{H}_{\mathcal{G}}) \subseteq \mathbf{WSym}$ . Let us prove the other inclusion. Let  $\varpi \in \mathcal{SP}$ , of cardinality  $k$  and of degree  $n$ . We define a graph  $P$  on  $[n]$  by the following property: for all  $i, j \in [n]$ , there is an edge between  $i$  and  $j$  if, and only if,  $i$  and  $j$  are in two different parts of  $\varpi$ . Note that  $\varpi \in \mathbb{IP}(G)$ . Let  $\varpi' \in \mathbb{IP}(G)$ . Necessarily, as the parts of  $\varpi'$  are independent subsets, each of them is included in a part of  $\varpi$ ; therefore,  $length(\varpi') \geq k$  with equality if, and only if,  $\varpi' = \varpi$ . Hence:

$$\mathbf{P}_{chr}(G) = W_{\varpi} + \text{a sum of } W_{\varpi'} \text{ with } length(\varpi') > k.$$

By a triangularity argument, we deduce that  $\mathbf{WSym} \subseteq \mathbf{P}_{chr}(\mathcal{H}_{\mathcal{G}})$ .

1. So  $\mathbf{WSym}$  is a Hopf subalgebra of  $\mathbf{WQSym}$ , isomorphic to a quotient of  $\mathcal{H}_{\mathcal{G}}$ , so is co-commutative.  $\square$

**Remark.** The Hopf algebra  $\mathbf{WSym}$ , known as the Hopf algebra of word symmetric functions, is described and used in [2, 4, 11]. Here is a description of its product and coproduct, with immediate notations:

- For any set partitions  $\varpi, \varpi'$  of respective degree  $m$  and  $n$ :

$$W_{\varpi}W_{\varpi'} = \sum_{\substack{\varpi'' \in \mathcal{SP}, \deg(\varpi'')=m+n, \\ \text{Pack}(\varpi''|_{[m]})=\varpi, \\ \text{Pack}(\varpi''|_{[m+n]\setminus[m]})=\varpi'}} W_{\varpi''}.$$

- For any set partition  $\varpi = \{P_1, \dots, P_k\}$ :

$$\Delta(P_w) = \sum_{I \subseteq [k]} W_{\text{Pack}(\{I_p | p \in I\})} \otimes W_{\text{Pack}(\{I_p | p \notin I\})}.$$

For example:

$$\begin{aligned} W_{\{1,2\}}W_{\{1\}} &= W_{\{1,2\},\{3\}} + W_{\{1,2,3\}}, \\ W_{\{1\},\{2\}}W_{\{1\}} &= W_{\{1\},\{2\},\{3\}} + W_{\{1,3\},\{2\}} + W_{\{1\},\{2,3\}}, \\ \Delta(W_{\{1,3\},\{2\},\{4\}}) &= W_{\{1,3\},\{2\},\{4\}} \otimes 1 + W_{\{1,3\},\{2\}} \otimes W_{\{1\}} + W_{\{1,2\},\{3\}} \otimes W_{\{1\}} \\ &\quad + W_{\{1\},\{2\}} \otimes W_{\{1,2\}} + W_{\{1,2\}} \otimes W_{\{1\},\{2\}} + W_{\{1\}} \otimes W_{\{1,2\},\{3\}} \\ &\quad + W_{\{1\}} \otimes W_{\{1,3\},\{2\}} + 1 \otimes W_{\{1,3\},\{2\},\{4\}}. \end{aligned}$$

### 3.4 Non-commutative version of $\phi_0$

**Proposition 29** *Let  $G$  be an indexed graph and  $f \in \mathbb{PC}(G)$ .*

1. *We define the equivalence  $\sim_f$  in  $V(G)$  as the unique one which classes are the connected components of the subsets  $f^{-1}(i)$ ,  $i \in [\max(f)]$ . Moreover, the coloring  $f$  induces a packed valid coloring  $\bar{f}$  of  $G/\sim_f$ :*

$$\forall i \in V(G), \bar{f}(\bar{i}) = f(i).$$

2. *Recall that  $G/\sim_f$  is an indexed graph; we denote its cardinality by  $k$ . We put:*

$$w_f = \bar{f}(1) \dots \bar{f}(k).$$

**Proof.** We have to prove that  $\bar{f}$  is a valid coloring of  $G/\sim_f$ . Let  $\bar{i}, \bar{j}$  be two vertices of  $G/\sim_f$ , related by an edge (this implies that they are different); we assume that  $\bar{f}(\bar{i}) = \bar{f}(\bar{j})$ . There exist  $i', j' \in V(G)$ , such that  $i' \sim_f i$  and  $j' \sim_f j$ , and  $i, j$  are related by an edge in  $G$ .



By definition of  $\sim_f$ , there exist vertices  $i' = i_1, \dots, i_k = i$ ,  $j = j_1, \dots, j_l = j'$  in  $G$  such that  $f(i_1) = \dots = f(i_k)$ ,  $g(j_1) = \dots = g(j_l)$ , and for all  $p, q$ ,  $i_p$  and  $i_{p+1}$ ,  $j_q$  and  $j_{q+1}$  are related by an edge in  $G$ . Hence, there is a path in  $G$  from  $i$  to  $j$ , such that for any vertex  $x$  on this path,  $f(i) = f(x) = f(j)$ : this implies that  $i \sim_f j$ , so  $\bar{i} = \bar{j}$ . This is a contradiction, so  $\bar{f}$  is valid.  $\square$

**Proposition 30** *Let us consider the following map:*

$$\Phi_0 : \begin{cases} \mathcal{H}_{\mathcal{G}} & \longrightarrow \mathbf{WSym} \\ G & \longrightarrow \sum_{f \in \mathbb{PC}(G)} w_f. \end{cases}$$

*This is a Hopf algebra morphism. Moreover, in  $E_{\mathcal{H}_{\mathcal{G}} \rightarrow \mathbf{WSym}}$ :*

$$\mathbf{P}_{chr} = \Phi_0 \leftarrow \lambda_{chr}.$$

**Proof.** Let  $G$  be an indexed graph. By proposition 29, we have a map:

$$\theta : \begin{cases} \mathbb{PC}(G) & \longrightarrow \bigsqcup \mathbb{PVC}(G/\sim) \\ f & \longrightarrow \bar{f} \in \mathbb{PVC}(G/\sim_f). \end{cases}$$

$\theta$  is injective: if  $\theta(f) = \theta(g)$ , then  $\sim_f = \sim_g$  and for any  $x \in V(G)$ ,

$$f(x) = \bar{f}(\bar{x}) = \bar{g}(\bar{x}) = g(x).$$

Let us show that  $\theta$  is surjective. Let  $\bar{f} \in \mathbb{PVC}(G/\sim)$ , with  $\sim \triangleleft G$ . We define  $f \in \mathbb{PC}(G)$  by  $f(x) = \bar{f}(\bar{x})$  for any vertex  $x$ . By definition of  $f$ , the equivalence classes of  $\sim$  are included in sets  $f^{-1}(i)$ , and are connected, as  $\sim \triangleleft G$ , so are included in equivalence classes of  $\sim_f$ : if  $x \sim y$ , then  $x \sim_f y$ . Let us assume that  $x \sim_f y$ . There exists a path  $x = x_1, \dots, x_k = y$  in  $G$ , such that  $f(x_1) = \dots = f(x_k)$ . So  $\bar{f}(\bar{x}_1) = \dots = \bar{f}(\bar{x}_k)$ . As  $\bar{f}$  is a valid coloring of  $G/\sim$ , there is no edge between  $\bar{x}_p$  and  $\bar{x}_{p+1}$  in  $G/\sim$  for any  $p$ ; this implies that  $\bar{x}_p = \bar{x}_{p+1}$  for any  $p$ , so  $x = x_1 \sim x_k = y$ . Finally,  $\sim = \sim_f$ , so  $\theta(f) = \bar{f}$ .

Using the bijection  $\theta$ , we obtain:

$$\begin{aligned} \Phi_0(G) &= \sum_{f \in \mathbb{PC}(G)} w_f \\ &= \sum_{\sim \triangleleft G} \sum_{\bar{f} \in \mathbb{PVC}(G/\sim)} w_{\bar{f}} \\ &= \sum_{\sim \triangleleft G} \mathbf{P}_{chr}(G/\sim) \\ &= \sum_{\sim \triangleleft G} \mathbf{P}_{chr}(G/\sim) \lambda_0(G|\sim) \\ &= (\mathbf{P}_{chr} \leftarrow \lambda_0)(G). \end{aligned}$$

Therefore,  $\Phi_0 = \mathbf{P}_{chr} \leftarrow \lambda_0$ , so is a Hopf algebra morphism, taking its values in  $\mathbf{WSym}$ . Moreover,  $\mathbf{P}_{chr} = \Phi_0 \leftarrow \lambda_0^{*-1} = \Phi_0 \leftarrow \lambda_{chr}$ .  $\square$

### 3.5 From non-commutative to commutative

For any  $k \geq 0$ , we denote by  $H_k$  the  $k$ -th Hilbert polynomial:

$$H_k(X) = \frac{X(X-1)\dots(X-k+1)}{k!}.$$

**Proposition 31** *The following map is a surjective Hopf algebra morphism:*

$$H : \begin{cases} \mathbf{WSym} & \longrightarrow \mathbb{Q}[X] \\ W_{\{I_1, \dots, I_k\}} & \longrightarrow H_k. \end{cases}$$

Moreover,  $H \circ \mathbf{P}_{chr} = P_{chr}$  and  $H \circ \Phi_0 = \phi_0$ .

**Proof.** For any graph  $G$ , for any integer  $k \geq 1$ :

$$P_{chr}(G)(k) = \sum_{f \in VPC(G)} \binom{\max(f)}{k} = \sum_{f \in VPC(G)} H_{\max(f)}(k) = H \circ \mathbf{P}_{chr}(G)(k).$$

So  $P_{chr} = H \circ \mathbf{P}_{chr}(G)$ . As  $\mathbf{P}_{chr}(\mathcal{H}_{\mathcal{G}}) = \mathbf{WSym}$ , this implies that  $H$  is a Hopf algebra morphism from  $\mathbf{WSym}$  to  $\mathbb{Q}[X]$ . For any  $x \in \mathcal{H}_{\mathcal{G}}$ , putting  $\delta(x) = x^{(1)} \otimes x^{(2)}$ :

$$\begin{aligned} H \circ \Phi_0(x) &= H \left( \mathbf{P}_{chr}(x^{(1)}) \lambda_0(x^{(2)}) \right) \\ &= H \circ \mathbf{P}_{chr}(x^{(1)}) \lambda_0(x^{(2)}) \\ &= P_{chr}(x^{(1)}) \lambda_0(x^{(2)}) \\ &= (P_{chr} \leftarrow \lambda_0)(x) \\ &= \phi_0(x). \end{aligned}$$

So  $H \circ \Phi_0 = \phi_0$ . □

**Remarks.**

1. We already observed that  $H : \mathbf{WQSym} \longrightarrow \mathbb{Q}[X]$  is a Hopf algebra morphism in [8].
2. In the commutative case,  $\phi_0$  is homogeneous and  $P_{chr}$  is not; in the non-commutative case,  $\Phi_0$  is not homogeneous, and  $\mathbf{P}_{chr}$  is.

## References

- [1] Eiichi Abe, *Hopf algebras*, Cambridge Tracts in Mathematics, vol. 74, Cambridge University Press, Cambridge-New York, 1980, Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka.
- [2] Nantel Bergeron, Christophe Reutenauer, Mercedes Rosas, and Mike Zabrocki, *Invariants and coinvariants of the symmetric groups in noncommuting variables*, Canad. J. Math. **60** (2008), no. 2, 266–296.
- [3] G. D. Birkhoff and D. C. Lewis, *Chromatic polynomials*, Trans. Amer. Math. Soc. **60** (1946), 355–451.
- [4] Jean-Paul Bultel, Ali Chouria, Jean-Gabriel Luque, and Olivier Mallet, *Word symmetric functions and the Redfield-Pólya theorem*, 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013), Discrete Math. Theor. Comput. Sci. Proc., AS, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2013, pp. 563–574.
- [5] Damien Calaque, Kurusch Ebrahimi-Fard, and Dominique Manchon, *Two interacting Hopf algebras of trees: a Hopf-algebraic approach to composition and substitution of B-series*, Adv. in Appl. Math. **47** (2011), no. 2, 282–308.
- [6] Alain Connes and Dirk Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Comm. Math. Phys. **199** (1998), no. 1, 203–242.

- [7] ———, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys. **210** (2000), no. 1, 249–273.
- [8] Loïc Foissy, *Commutative and non-commutative bialgebras of quasi-posets and applications to Ehrhart polynomials*, arXiv:1605.08310, 2016.
- [9] David D. Gebhard and Bruce E. Sagan, *A chromatic symmetric function in noncommuting variables*, J. Algebraic Combin. **13** (2001), no. 3, 227–255.
- [10] Frank Harary, *Graph theory*, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.
- [11] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon, *Commutative combinatorial Hopf algebras*, J. Algebraic Combin. **28** (2008), no. 1, 65–95.
- [12] Christian Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995.
- [13] Dominique Manchon, *On bialgebras and Hopf algebras or oriented graphs*, Confluentes Math. **4** (2012), no. 1, 1240003, 10.
- [14] J.-C. Novelli and J.-Y. Thibon, *Polynomial realization of some trialgebras*, FPSAC’06 (San Diego) (2006).
- [15] Mercedes H. Rosas, *MacMahon symmetric functions, the partition lattice, and Young subgroups*, J. Combin. Theory Ser. A **96** (2001), no. 2, 326–340.
- [16] Gian-Carlo Rota, *On the foundations of combinatorial theory. I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368.
- [17] William R. Schmitt, *Incidence Hopf algebras*, J. Pure Appl. Algebra **96** (1994), no. 3, 299–330.
- [18] Richard P. Stanley, *Acyclic orientations of graphs*, Discrete Math. **5** (1973), 171–178.
- [19] Moss E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.